Holographic fluids: gravity and quantum mechanics

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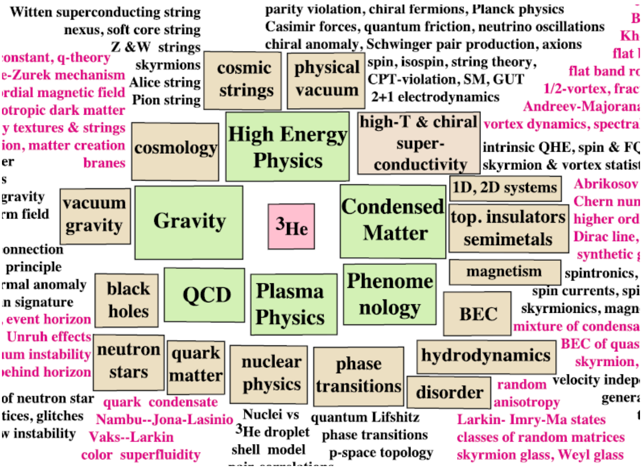
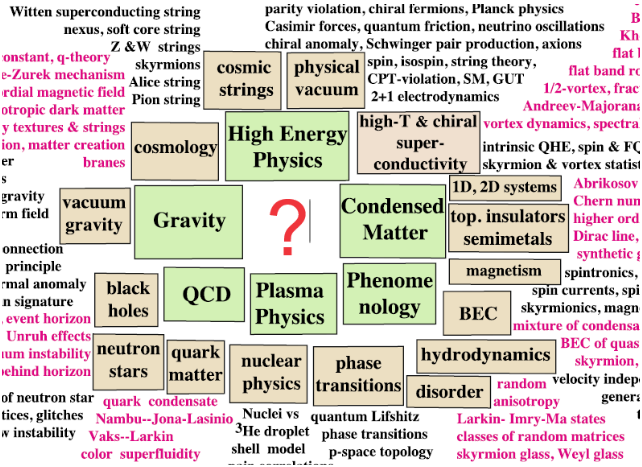
Content

^{3}He Universe

A recent view with trimmed boundaries.



From Volovik (JLTP, 2021).



Inflational expansion (2004-...).

Universal dissipative dynamics?

Non-equilibrium thermodynamics

- Carl Henry Eckart (1940, 1948). Eckart instability. Stepfather of the discipline.
  [PhD: Caltech (1925-7), matrix-wave, 1926, postdoc in Munich at Sommerfeld (1928), Uni. Chicago (1928-41) Fermi's bombtheory group with Wigner, anti-bomb, (sabbaticals in Princetown), UCSD (1946-73), Marine PL, underwater sound, Klara Dán von Neumann (1958-)).]
- Variational principles of different kind? Helmholtz conditions. Several attempts. My conclusion: too many different solutions.
- Second law compatibility of material properties. Internal variables. Memory and nonlocal effects.

A general construction mechanism of evolution equations emerges.
Compatibility with mechanics?
Compatibility with ideal, non-dissipative laws?

What is thermodynamics?

Fundamental or emergent?

- Statistical physics is special, thermodynamics is general.
- Separation of universal from particular.
- Second Law is general, there are statistical demonstrations.
- Second Law can be applied for fields.

Thermodynamics is a stability theory.
(📖T. Matolcsi, 📖W.M. Haddad, 📖VP (PTRSA 2023))

Are there some original, genuine consequences??

Local, nonlocal and weakly nonlocal

Locality in space(time)

- Local fields and local field equations. Example: $\varphi(t, \mathrm{x})$, Poisson equation.
- Space integrals in the field equations: strong nonlocality.
- Nonlocal fields: Example: $f(t, \mathrm{x}_1, \mathrm{x}_2)$, Liouville equation, entanglement.
- Weak nonlocality: extension of the field equations with higher order space derivatives. Example: gradient fluids, Horndeski gravity.

Locality in time

- Locality in time. No memory. Markov process.
- Memory functionals in the field equations: strong memory. Example: principle of fading memory.
- Weak 'nonlocality' in time: higher order time derivatives in the field equations. Example: second sound, delay and inertia.

Temporal nonlocality and spatial locality are interdependent.
Action at a distance: vacuum solution of a local theory.

Holography

Holography $\leftarrow$ holos+graphe = complete, whole + drawing, writing.

Optical holography

- Dennis Gábor. Reproduction of 3 dimensional information from 2 dimensional projections.
- Interferometric. Amplitude and phase. For any wavelike propagation. E.g. ambisonic sound.

Holography in quantum field theories

- Generalisation of black hole thermodynamics. Hawking, t'Hooft, Susskind. Entropy is area.
- Abstracted in string theory. Expected in quantum gravity.
- AdS-CFT correspondence.

Holographic principle + Unruh effect $\Rightarrow$ field equation of gravity (Newtonian and GR)

📖Jacobson (PRL 1995), ..., 📖Verlinde (JHEP 2011)

Classical holography I

Newtonian gravity ($\Delta\varphi = 4\pi G\rho$):

$$\rho\nabla\varphi = \nabla\cdot\boldsymbol{P}_{grav}(\nabla\varphi) = \nabla\cdot\left(\frac{1}{4\pi G}\left[\nabla\varphi\nabla\varphi - \frac{1}{2}(\nabla\varphi)^2\boldsymbol{I}\right]\right)$$

Maxwell stress tensor.

Euler fluids are holographic

Ideal Euler fluids: $\boldsymbol{P}_{Euler} = p(u,\rho)\boldsymbol{I}$. p is the thermostatic pressure, e.g. ideal gas.

$$\nabla\cdot\boldsymbol{P}_{Euler} = \nabla p = \rho\nabla\mu + \rho s\nabla T$$

Follows from the Gibbs-Duhem relation: $0 = sdT - vdp + d\mu$. For isothermal processes of ANY fluid the chemical potential is a mechanical potential. Friedmann equation.

Classical holographic property:

$$\nabla\cdot\boldsymbol{P}(...) = \rho\nabla\phi(...)$$

Constitutive (...), material property. Thermodynamics or field equation dependent?

Entropy inequality

Pure dissipation: heat conduction in CIT

CIT = Classical Irreversible Thermodynamics.
The calculation of entropy production (Eckart, 1940):

$$\rho\dot{e} + \nabla \cdot \boldsymbol{q} = 0, \qquad de = Tds$$

$$\rho\dot{s} + \nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) = \frac{\rho\dot{e}}{T} + \nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) = \boxed{\boldsymbol{q} \cdot \nabla \frac{1}{T} \geq 0}$$

Solution of the inequality: $\boldsymbol{q}(e, \nabla e) = \lambda_T \nabla \frac{1}{T} = -\lambda \nabla T$, $\lambda(e) \geq 0$.

General aspects:

- spacetime: comoving derivative, constitutive state space,
- entropy density: local thermodynamic potential,
- entropy inequality.

Material properties (statistical and kinetic origin?):

- static EOS: $e = cT$,
- constitutive EOT: $\boldsymbol{q} = -\lambda(e)\nabla T(e)$.

Remark I. Nonrelativistic balances

Balance of momentum. Global form:

$$\dot{M} = -F_{surf} + F_{bulk}.$$

Local form and substantial forms:

$$\rho\dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P} = -\rho\nabla\varphi, \qquad \rho\dot{v}^i + \partial_k P^{ik} = -\rho\partial^i\varphi. \tag{1}$$

Bulk and surface forces. Substantial or comoving derivative, Convective and conductive current densities, Hidden Galilean covariance.

Particle or field??

$$\rho\dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P}_{grav} = 0 \quad \Longleftrightarrow \quad \dot{\boldsymbol{v}} = -\nabla\varphi$$

Test particle and integrating screens. Constant background field or field theory?
($\dot{\rho} + \rho\nabla \cdot \boldsymbol{v} = 0$, $\Delta\varphi = 4\pi G\rho$)
Newtonian form:

$$\dot{M} = F$$

The universality of point mass modell.

Remark II. Galilean relativistic balances

Spacetime aspects - separation of material and motion

$$\frac{\partial(\rho e_{total})}{\partial t} + \nabla \cdot (\boldsymbol{q} + \rho \boldsymbol{v} e_{total} + \boldsymbol{P} \cdot \boldsymbol{v}) = 0, \quad \rightarrow \quad \rho \dot{e}_{total} + \nabla \cdot (\boldsymbol{q} + \boldsymbol{P} \cdot \boldsymbol{v}) = 0$$

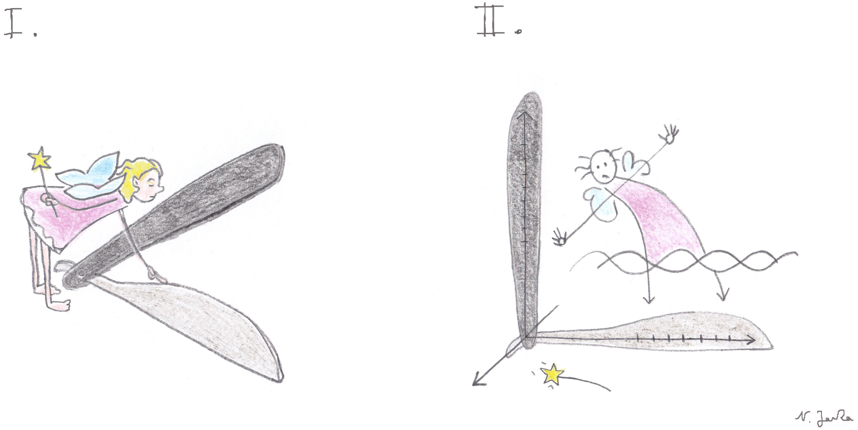
Relative form of a four-divergence:

- Galilean four-vector: $(\rho e, \boldsymbol{q})$, convective and conductive current densities.
- Comoving(substantial) time derivative: $\dot{e} = \frac{\partial e}{\partial t} + \boldsymbol{v} \cdot \nabla e$. Transformation rule for a timelike covector.
- ∇e is spacelike covector: does not transform.
- total and internal energies: $e = e_{TOT} - v^2/2$.

Consequences

- What is comoving? Mass? Energy? Observer representations. Flow-frames.
- Total energy, kinetic energy and internal energy. Galilean relativistic energy-momentum-mass four-tensor. Entropy production is objective.
- Temperature is a Galilean relativistic four-vector: thermal reference frames.

Ockham's razor



"Is there a harmony of mathematics and physics??"

Constitutive state space

Coleman-Noll and Liu procedures. Separation of functions and variables.
The entropy inequality is *conditional*:

$$\rho \dot{e} + \nabla \cdot \boldsymbol{q}(e, \nabla e) = 0,$$
$$\rho \dot{s}(e, \nabla e) + \nabla \cdot \boldsymbol{J}(e, \nabla e) - \Lambda(e, \nabla e)(\rho \dot{e} + \nabla \cdot \boldsymbol{q}(e, \nabla e)) =$$
$$\rho \frac{\partial s}{\partial \nabla e} (\nabla e)^{\cdot} + \rho \left(\frac{1}{T} - \Lambda \right) \dot{e} + ... \geq 0$$

Liu-procedure, Lagrange–Farkas-multipliers. It follows that:

$$\frac{\partial s}{\partial \nabla e}(e, \nabla e) = 0, \qquad \Lambda = \frac{1}{T}, \quad \text{and} \quad \boldsymbol{q}(e, \nabla e) \cdot \nabla \left(\frac{1}{T}(e) \right) \geq 0$$

Constitutive state variables: $(e, \nabla e)$
$\rightarrow$ thermodynamic state variables: (e)
Process direction variables: $(\dot{e}, (\nabla e)^{\cdot}, \nabla^2 e)$

Weakly nonlocal extensions

Classified by constitutive state spaces and constraints

- Fluid mechanics. Mass, velocity and energy. $(\rho, \nabla\rho, v, \nabla v, \ e, \nabla e)$
  Constraints: balances of mass, momentum and energy ($\rightarrow$ quantum mechanics and more)
  $\rightarrow$ Fourier-Navier-Stokes equations.
- Fluid mechanics + scalar field $(\rho, \nabla\rho, v, \nabla v, e, \nabla e, \varphi, \nabla\varphi, \ \nabla^2\varphi)$
  Constraint: evolution equation, balances of mass momentum and energy.
  $\rightarrow$ Fourier-Navier-Stokes + Newtonian gravity and more
- Fluid mechanics + second order weak nonlocality in density. Mass, velocity and energy. $(\rho, \nabla\rho, \nabla^2\rho, v, \nabla v, \ e, \nabla e)$
  Constraints: balances of mass, momentum and energy
  $\rightarrow$ Korteweg fluids, superfluids, quantum mechanics and more

Newtonian gravity

📖VP-Abe (Physica A, 2022)
📖Abe-VP (Symmetry, 2022)
📖Pszota-VP (arXiv: 2306.01825)

Scalar field and hydrodynamics

$s(e - \varphi - \frac{\nabla\varphi\cdot\nabla\varphi}{8\pi G\rho}, \rho)$. Specific Gibbs relation:

$$du = Tds + \frac{p}{\rho^2}d\rho = de - d\left(\varphi + \frac{\nabla\varphi\cdot\nabla\varphi}{8\pi G\rho}\right).$$

The potential energy, φ, the field energy and internal energy are separated.

Balances of mass, momentum, internal energy + field equation:

$$\dot\rho + \rho\nabla\cdot\boldsymbol{v} = 0,$$
$$\rho\dot{\boldsymbol{v}} + \nabla\cdot\boldsymbol{P} = \boldsymbol{0},$$
$$\rho\dot e + \nabla\cdot\boldsymbol{q} = -\boldsymbol{P} : \nabla\boldsymbol{v},$$
$$\dot\varphi = f.$$

Constraints of the entropy inequality:

$$\boxed{\rho\dot s + \nabla\cdot\boldsymbol{J} = \Sigma \geq 0}$$

Gravity

Constitutive state variables: $(e, \nabla e, \rho, \nabla\rho, (\boldsymbol{v}), \nabla\boldsymbol{v}, \varphi, \nabla\varphi, \nabla^2\varphi)$

$\rightarrow$ thermodynamic state variabless: $(e, \rho, \varphi, \nabla\varphi)$

$$\rho\dot{s} + \nabla\cdot\boldsymbol{J} =$$

$$\left(\boldsymbol{q} + \frac{\dot{\varphi}}{4\pi G}\nabla\varphi\right)\cdot\nabla\left(\frac{1}{T}\right)$$

$$+\boxed{\frac{f}{4\pi G T}\left(\Delta\varphi - 4\pi G\rho\right)}$$

$$-\left[\boldsymbol{P} - p\boldsymbol{I} - \frac{1}{4\pi G}\left(\nabla\varphi\nabla\varphi - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\boldsymbol{I}\right)\right] : \frac{\nabla\boldsymbol{v}}{T} \geq 0$$

- Perfect self-gravitating (isothermal) fluids are holographic:

$$\nabla\cdot\left(p\boldsymbol{I} + \frac{1}{4\pi G}\left(\nabla\varphi\nabla\varphi - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\boldsymbol{I}\right)\right) = \rho\nabla(\mu+\varphi)$$

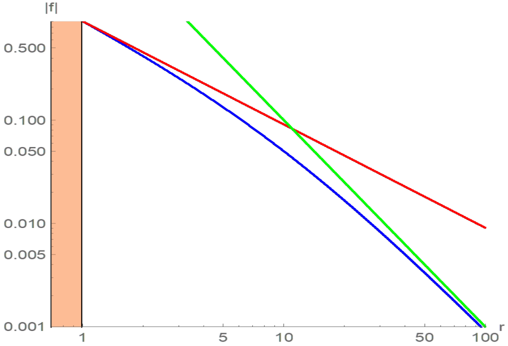
Nonlinear extension, static, nondissipative field

Stationary nondissipative field equation:

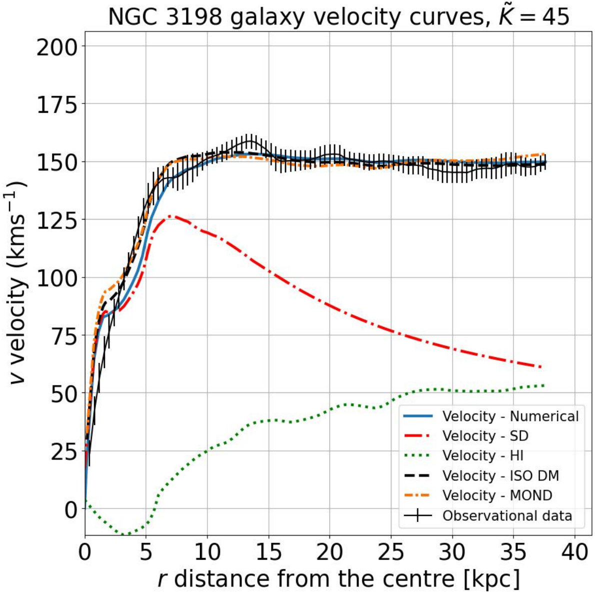
$$\boxed{0 = \Delta\varphi - 4\pi G\rho - K\nabla\varphi\cdot\nabla\varphi.}$$

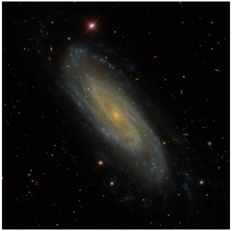
Spherical symmetric force field. Crossover. Apparent and real masses:

$$f(r) = -\frac{r_1}{Kr(r+r_1)} = -\frac{GM_{aa}}{r(r+r_1)}$$



Thermodynamic gravity, MOND and Dark Matter





NGC 3198	
M_{DM+BM}	M_{aa}
190	110

Unit: $10^9 M_\odot$

Thanks to M. Pszota

Korteweg fluids

📖VP-Fülöp (Proc. Roy. Soc., 2004)
📖VP-Kovács (Phil. Trans. Roy. Soc. A, 2020)
📖VP (Physics of Fluids, 2023)

Korteweg fluids: history

Van der Waals: gradient of density is a thermodynamic variable. Surface tension and capillarity.
Korteweg (1905): second gradient of density, pressure expansion.

Balances of mass, momentum and internal energy:

$$\dot{\rho} + \rho \nabla \cdot \boldsymbol{v} = 0,$$
$$\rho \dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P} = \boldsymbol{0},$$
$$(\rho \dot{e} + \nabla \cdot \boldsymbol{q} = -\boldsymbol{P} : \nabla \boldsymbol{v}.)$$

$$\boldsymbol{P} = \left(p - \alpha \Delta \rho - \beta (\nabla \rho)^2\right) \boldsymbol{I} - \delta \nabla \rho \circ \nabla \rho - \gamma \nabla^2 \rho$$

$\alpha, \beta, \gamma, \delta$ are density dependent material parameters.

Instable. Second law? Eckart fluids 1940, Dunn and Serrin (ARMA, 1985).

Korteweg fluids – Liu procedure

Constitutive state variables: $(e, \nabla e, \rho, \nabla\rho, \nabla^2\rho, (\boldsymbol{v}), \nabla\boldsymbol{v})$

$\rightarrow$ thermodynamic state variables: $(e, \rho, \nabla\rho)$

Process direction: $(\dot{e}, (\nabla e)^{\cdot}, \nabla^2 e, \dot{\rho}, (\nabla\rho)^{\cdot}, (\nabla^2\rho)^{\cdot}, \nabla^3\rho, \dot{\boldsymbol{v}}, (\nabla^2\boldsymbol{v})^{\cdot})$

$$\rho\dot{s} + \nabla\cdot\boldsymbol{J} = \boldsymbol{q}\cdot\nabla\left(\frac{1}{T}\right) -$$

$$-\left[\boldsymbol{P} - p\boldsymbol{I} - \frac{\rho^2}{2}\left(\nabla\cdot\frac{\partial s}{\partial\nabla\rho}\boldsymbol{I} + \nabla\frac{\partial s}{\partial\nabla\rho}\right)\right] : \frac{\nabla\boldsymbol{v}}{T} \geq 0$$

- Rigorous methods are essential.
- The pressure of an ideal, non-dissipative Korteweg fluid is:

$$\boxed{\boldsymbol{P} = p(e,\rho)\boldsymbol{I} + \frac{\rho^2}{2}\left(\nabla\cdot\frac{\partial s}{\partial\nabla\rho}\boldsymbol{I} + \nabla\frac{\partial s}{\partial\nabla\rho}\right)}$$

Perfect Korteweg fluids are holographic

$$\boldsymbol{P}_K = \frac{\rho^2}{2}\left(\nabla \cdot \frac{\partial s}{\partial \nabla\rho}\boldsymbol{I} + \nabla \frac{\partial s}{\partial \nabla\rho}\right)$$

- Classical holographic property, with internal energy:

$$\boxed{\nabla \cdot \boldsymbol{P}_K = \rho(\nabla\phi + T\nabla s)}, \quad \text{where} \quad \phi = \frac{\partial \rho u}{\partial \rho} - \nabla \cdot \frac{\partial(\rho u)}{\partial \nabla\rho} = \delta_\rho(\rho u)|_{\rho s}$$

Functional derivative. Isothermal and adiabatic potentials.

- Momentum balance: continuum AND point mass

$$\rho\dot{\boldsymbol{v}} + \nabla \cdot \boldsymbol{P}_K = \rho(\dot{\boldsymbol{v}} + \nabla\phi) = 0 \quad \rightarrow \quad \dot{\boldsymbol{v}} = -\nabla\phi$$

- Conserved vorticity follows.
- $\phi(\rho, \nabla\rho, ...)$: Bohm potential, a chemical potential of superfluids.
- How can we get the Schrödinger equation?

Quantum to hydro

Schrödinger (superfluid) equation:

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\Delta\psi - V\psi = 0$$

Madelung transformation:

$$\psi = Re^{i\varphi},$$

ρ density (probability or superfluid), $R=\sqrt{\rho}$, φ velocity potential: $\boldsymbol{v}=\frac{\hbar}{m}\nabla\varphi$.

$$\frac{i\hbar}{2\rho}\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})\right)\psi-\left(m\frac{\hbar}{m}\frac{\partial\varphi}{\partial t}+m\frac{v^2}{2}-\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}-V\right)\psi=0$$

Continuity and Bernoulli equations of classical rotation free fluids. The gradient of the second one

$$\dot{\boldsymbol{v}}+\nabla(U_Q+V)=0$$

$U_Q(\rho,\nabla\rho,\nabla^2\rho)=-\frac{\hbar^2}{2m^2}\frac{\Delta R}{R}$ is the Bohm potential. $V(\ldots)$ may be the chemical potential of boson interactions. Ginzburg-Gross-Pitaevskii-Sobaynin-... theories.
What is the origin of Bohm potential?

Probabilistic Korteweg fluids – additivity

Zeroth Law of thermodynamics: separability of independent physical systems.
Multicomponent normal fluids. Notation: $\rho_1 = \rho_1(\boldsymbol{x}_1)$.

$$u(\rho_1 + \rho_2) = u(\rho_1) + u(\rho_2).$$

Multicomponent probabilistic fluids:

$$u(\rho_1 \rho_2) = u(\rho_1) + u(\rho_2).$$

Functional condition, $\rho_{tot} = \rho_1 \rho_1$:

$$u(\rho_{tot}, (\nabla \rho_{tot})^2) = u\left(\rho_1 \rho_2, (\rho_2 \nabla_1 \rho_1)^2 + (\rho_1 \nabla_2 \rho_2)^2\right) =$$
$$u(\rho_1, (\nabla_1 \rho_1)^2) + u(\rho_2, (\nabla_2 \rho_2)^2).$$

Unique solution:

$$\boxed{u(\rho, (\nabla \rho)^2) = k \ln \rho + \frac{\kappa}{2} \frac{(\nabla \rho)^2}{\rho^2}}$$

Independent Schrödinger equations for independent particles/components.

QFT, GR can be fluids: 🕮Jackiw et al. (JP A, 2004), 🕮Biró-VP(FP, 2015), ...

Quantum versus hydro

Analogy :

$$\boxed{\text{quantum mechanics}} \rightarrow \boxed{\begin{array}{c}\textit{superfluids}\\ \textit{and}\\ \textit{more}\end{array}} \rightarrow \boxed{\text{capillary fluids and more}}$$

Deduction and universality :

$$\boxed{\begin{array}{c}\text{gradient fluids}\\ \text{and/or}\\ \text{fields}\end{array}} \Longrightarrow \boxed{\begin{array}{c}\textit{superfluids}\\ \textit{and}\\ \textit{more}\end{array}} \Longrightarrow \boxed{\text{quantum mechanics}}$$

Second Law of Thermodynamics $\Longrightarrow$ Classical holography.

Additivity $\Longrightarrow$ Bohm potential

Concepts, practical tools, mechanisms and foundations

Concepts: Quantum plasmas, plasmons

Correlated many-particle systems. 📖Jánossy et al. (APH, 1963-77)

Aspects of quantum hydrodynamics

- Time dependent Hartree equations can be converted to MQHD - microscopic QHD, i.e. multicomponent qfluid equations.
- Various partial averages e.g. standard vs. many-fermion Bohm potentials:

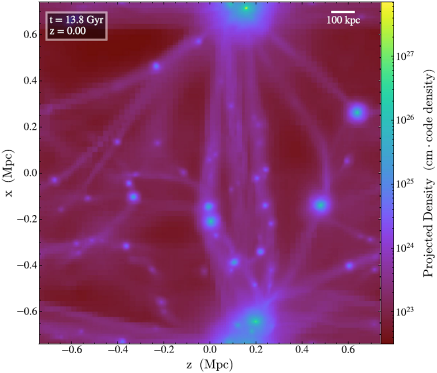
$$\phi_B = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \quad \rightarrow \quad \tilde{\phi}_B = -\frac{\hbar^2}{2mN}\sum_{i=1}^{N} f_i \frac{\Delta\sqrt{\rho_i}}{\sqrt{\rho_i}}$$

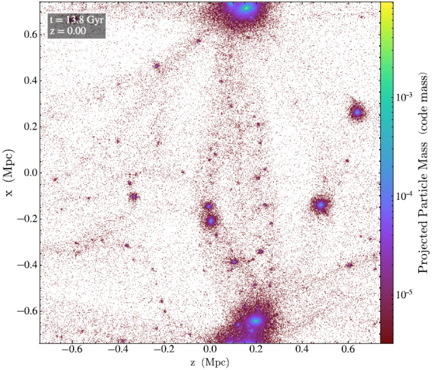
  ρ - mean electron density, N - total number of electrons, f_i occupation number given by a Fermi-function, n_i orbital probability density,
- Mixed states are not problematic with modified initial sampling probability: $\sum_i f_i \rho_i(t = 0, \boldsymbol{R})$.

Many continuity and averaged momentum: equivalent to time dependent DFT (Kohn-Sham equations). Exchange potential.

Practical tool: SFDM (Scalar Field Dark Matter)

– Correlated many-boson systems. Bose-Einstein condensates, superfluids. In cosmology: 🕮Ruffini and Bonazzola (PR 1969).

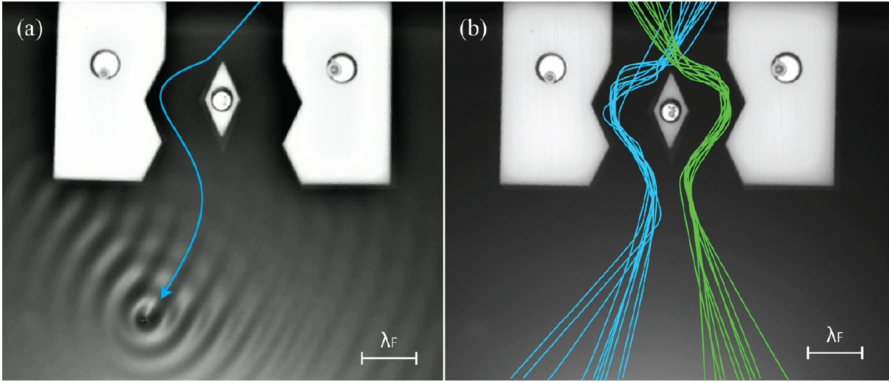




– Evolution of ΛCDM halos with hydrodynamics and N-body simulations.
– Solution of several small scale problems of ΛCDM, like "core-cusp", "too-big-to-fail", etc...
🕮Foidl et al. (PRD 2023) 🕮Shapiro et al. (MNRAS 2021)

Insight and mechanism: pilot-wave hydrodynamics.

Bohmian mechanism: double solution pilot-wave hydrodynamics 🕮de Broglie (1956,1987)



Bouncing droplets over an excited surface. 🕮 Coulder et al. (Nature, 2005), 🕮 Bush (ARFM, 2015), 🕮 Frumkin et al. (PRA, 2022).
Stochastic "interpretation": 🕮 Fényes (ZfP, 1952)
Stochastic electrodynamics: 🕮 de la Pena-Cetto (1992)

More than six worlds since 1992 : 📖Bell (FP, 1992)

Interpretation : inferior by nomination. Original $\geq$ interpretation.
Bohm vs. de Broglie today 📖Drezet (special issue of FP, 2023)

What are the Bohmian benchmarks ?

- theory : predictions beyond quantum mechanics ?
- origin of Bohm potential ?
- spin (empty waves, fermions) ?
- classical-quantum transition ?
- measurement, relativistic, ...

Real or not ?

- Deus ludens. Probabilistic is real.
- Are particles solitons ? Not really.
- What is the background mechanism ? Universality : anything. Level stability ?
- Spacetime aspects, operators, Wigner functions, ...

Summary

Emergent classical holography and emergent evolution.

- The Second Law of Thermodynamics is applicable for fields and informative in the marginal case of zero dissipation.
- Variational principles are not necessary.
- The Second Law of Thermodynamics is (looks like) fundamental.

Case 1: There is a thermodynamic road to gravity.

$$\text{Fluid} + \text{scalar internal variable} \longrightarrow \text{gravity}$$

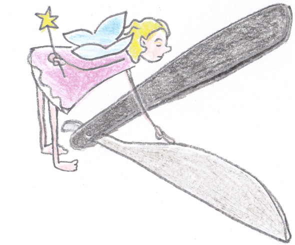
- Second law with zero dissipation $\Longrightarrow$ classical holography
- Energy type, quadratic $\Longrightarrow$ gravity

Case 2: There is a thermodynamic road to quantum physics.

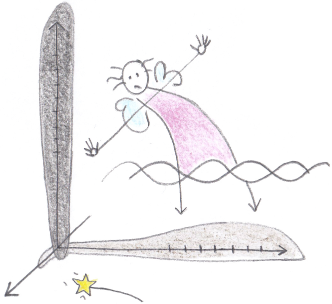
$$\text{Korteweg fluids} \longrightarrow \text{quantum mechanics}$$

- Second law with zero dissipation $\Longrightarrow$ classical holography
- Additivity $\Longrightarrow$ quantum systems

I.



II.



"This may be true, because it is mathematically trivial."

(somebody from Princeton, according to R. Pisarski)

Thank you for the attention!

Variational principles for dissipative processes

Condition: symmetry

$$\boxed{\hat{\Theta}(\varphi) = 0}, \qquad \exists F : Dom(\hat{\Theta}) \to \mathbb{R}, \quad \delta F(\varphi) = \hat{\Theta}(\varphi)$$

δ derivation in a Banach (or Frechet) spaces, boundary conditions, ...

Necessary condition: $\hat{\Theta}$ is symmetric.

Many different variational principles

- Potentials: $\hat{\Theta} \circ \hat{\varphi}(\varphi) = 0$, where $\hat{\Theta} \circ \hat{\varphi}$ is symmetric
- Integrating multipliers: $\hat{T} \circ \hat{\Theta} = 0$, where $\hat{T} \circ \hat{\Theta}$ is symmetric
- Change the operator: $(\hat{\Theta}(\varphi))^2 = 0$, and neglect parts
- Change the function space: Gyarmati principle, ...,

All of them are right, which one is the true?

Farkas' lemma

Liu procedure: linear algebra and analysis.

Farkas(-Minkowski-Haar) lemma

If $\boldsymbol{a}_i \neq \boldsymbol{0}$, $i = 1, ..., n$ are vectors of a finite dimensional vector space $\boldsymbol{V}$ and $S = \{\boldsymbol{p} \in \boldsymbol{V^*} | \boldsymbol{p} \cdot \boldsymbol{a}_i \geq 0, i = 1, ..., n\}$ is a subset of the dual vector space, then the following staments are equivalent for all $\boldsymbol{b} \in \boldsymbol{V}$ vectors:
(i) $\boldsymbol{p} \cdot \boldsymbol{b} \geq 0$ for all $\boldsymbol{p} \in S$.
(ii) There exist $\lambda_1, ..., \lambda_n$ nonnegative real numbers, so that $\boldsymbol{b} = \sum_{i=1}^n \lambda_i \boldsymbol{a}_i$.
Remark:

$$\boldsymbol{p} \cdot \boldsymbol{b} - \sum_{i=1}^n \lambda_i \boldsymbol{p} \cdot \boldsymbol{a}_i = \boldsymbol{p} \cdot (\boldsymbol{b} - \sum_{i=1}^n \lambda_i \cdot \boldsymbol{a}_i) \geq 0, \quad \forall \boldsymbol{p} \in \boldsymbol{V}^*.$$

History:
Farkas proved it in his analysis of Fourier principle (of mechanics) in 1895 in Hungarian. Minkowski and Haar provided independent proofs later. The lemma is the base of the Bell inequalities and also the Karush-Kuhn-Thucker theorems of optimisation.

Phase field evolution equations, variational derivatives

Scalar field evolution: $\dot{\varphi} = f(\varphi, \nabla\varphi)$

$$\dot{S}(\varphi, \nabla\varphi) - \lambda(\dot{\varphi} - f(\varphi, \nabla\varphi)) = \underline{(\partial_\varphi S - \lambda)}\dot{\varphi} + \underline{\partial_{\nabla\varphi} S}\, \nabla\dot{\varphi} + \lambda f \geq 0$$

$$\partial_\varphi S - \lambda = 0, \quad \partial_{\nabla\varphi} S = 0,$$

$$\boxed{0 \leq f\, \partial_\varphi S} \quad \rightarrow \quad f = l\, \partial_\varphi S, \quad (l \geq 0)$$

Extended approach: $\dot{\varphi} = f(\varphi, \nabla\varphi, \nabla^2\varphi)$

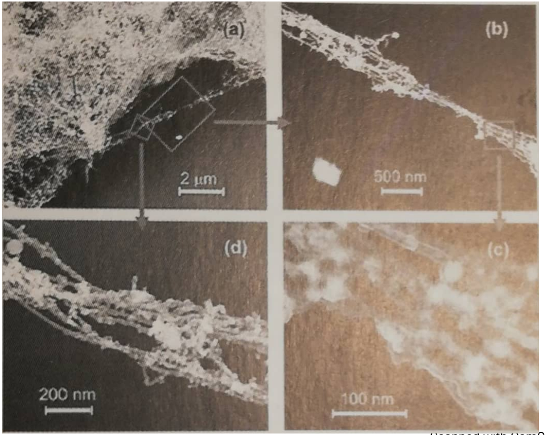
- Higher order state space: $(\varphi, \nabla\varphi, \nabla^2\varphi)$;
- Constitutive entropy flux;
- Gradient constraints: $\nabla\dot{\varphi} = \nabla f$

$$\dot{S} + \nabla J - \lambda(\dot{\varphi} - f) - \Lambda(\nabla\dot{\varphi} - \nabla f) \geq 0$$

$$\partial_\varphi S = \lambda, \quad \partial_{\nabla\varphi} S = \Lambda^i, \quad \partial_{\partial\varphi} S = 0$$

$$J = -\partial_{\nabla\varphi} S\, f + \hat{J}(\varphi, \nabla\varphi) \qquad \boxed{0 \leq f\,(\partial_\varphi S - \nabla(\partial_{\nabla\varphi} S))} = f\frac{\delta S}{\delta\varphi}$$

Quantum fluids exist:



Vortex lines in He II, from boundary to boundary. Donelly, 1991.

Superfluidity

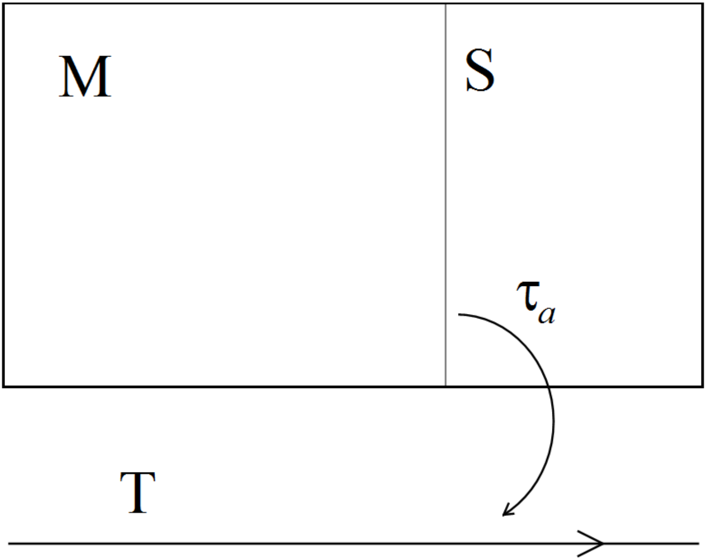
Histocical remarks and more

- Landau, ZhETP (1941).
- Bohm and Pines (1952), Gross and Pitaevskii (1961).
- Dense plasmas. Critical survey of Bonitz et al. (2019).
- QGP, T. Kodama et al., stochastic qm., Jackiw et al (2004).
- G.E. Volovik: The Universe in a Helium Droplet (2003).

Theory notes: there is no quantisation

- There is a Hamiltonian but there is no Lagrangian.
- Hydro is an effective theory and cannot be quantised.
- (Note: Like gravity. Therefore gravity cannot be quantized, q-gravity does not exist.)

The four dimensions of Galilean relativistic space-time

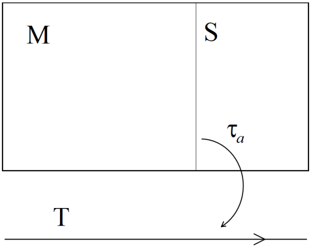


Mathematical structure of Galilean relativistic space-time

1. The *space-time* $\mathbb{M}$ is an oriented four dimensional vector space of the $x^a \in \mathbb{M}$ *world points or events*. There are no Euclidean or pseudoeuclidean structures on $\mathbb{M}$: the length of a space-time vector does not exist.
2. The *time* $\mathbb{I}$ is a one dimensional oriented vector space of $t \in \mathbb{I}$ *instants*.
3. $\tau_a : \mathbb{M} \to \mathbb{I}$ is the *timing* or *time evaluation*, a linear surjection.
4. $\delta_{\bar{a}\bar{b}} : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \otimes \mathbb{R}$ Euclidean structure is a symmetric bilinear mapping, where $\mathbb{E} := Ker(\tau) \subset \mathbb{M}$ is the three dimensional vector space of *space vectors*.

- Simplification: space-time and time are affine spaces
- Simplification: measure lines.
- Abstract indexes: $a, b, c, ...$ for $\mathbb{M}$, $\bar{a}, \bar{b}, \bar{c}, ...$ for S

Vectors an covectors are different



$$\boxed{A'^{\alpha}B'_{\beta} = A^{\alpha}B_{\beta} = AB + A^i B_i}$$

$$\begin{pmatrix} t' \\ x'^i \end{pmatrix} = \begin{pmatrix} t \\ x^i + v^i t \end{pmatrix}$$

Vector transformations (extensives):

$$\begin{pmatrix} A' \\ A'^i \end{pmatrix} = \begin{pmatrix} A \\ A^i + v^i A \end{pmatrix}$$

Covector transformations (derivatives):

$$\begin{pmatrix} B' & B'_i \end{pmatrix} = \begin{pmatrix} B - B_k v^k & B_i \end{pmatrix}$$

Balances: absolute, local and substantial

$$\boxed{\partial_a A^a = 0} \qquad \longrightarrow \qquad \begin{aligned} u^a: \quad D_u A + \partial_i A^i &= d_t A + \partial_i A^i = 0, \\ u'^a: \quad D_{u'} A + \partial_i A'^i &= \partial_t A + \partial_i A'^i = 0. \end{aligned}$$

$(a,b,c \in \{0,1,2,3\})$

Transformed: $(d_t - v^i\partial_i)A + \partial_i(A^i + Av^i) = d_t A + A\partial_i v^i + \partial_i A^i = 0$

Mass, energy and momentum

What kind of quantity is the energy?

- Square of the relative velocity: 2nd order tensor
- Kinetic theory: trace of a contravariant second order tensor.
- Energy density and flux: additional order

Basic field:

$$\boxed{Z^{abc} = z^{bc}u^a + z^{abc} : \qquad \text{mass-energy-momentum density-flux tensor}}$$

$a, b, c \in \{0,1,2,3\}, \quad \bar{a}, b, c \in \{1,2,3\}$

$$z^{bc} \to \begin{pmatrix} \rho & p^b \\ p^c & e^{bc} \end{pmatrix}, \qquad z^{\bar{a}bc} \to \begin{pmatrix} j^{\bar{a}} & P^{ab} \\ P^{ac} & q^{\bar{a}\bar{b}\bar{c}} \end{pmatrix}, \qquad e = \frac{e^b{}_b}{2}$$

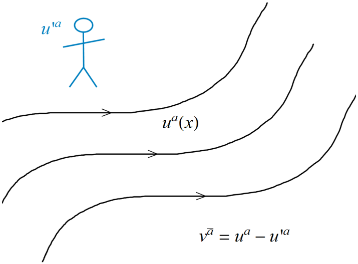
Galilean transformation

$$\boxed{Z'^{\alpha\beta\gamma} = G^{\alpha}_{\mu} G^{\beta}_{\nu} G^{\gamma}_{\kappa} Z^{\mu\nu\kappa}}$$

$$Z^{\alpha\beta\gamma} = \left(\begin{pmatrix} \rho & p^i \\ p^j & e^{ji} \end{pmatrix} \quad \begin{pmatrix} j^k & P^{ki} \\ P^{kj} & q^{kij} \end{pmatrix} \right), \quad G^{\alpha}_{\nu} = \begin{pmatrix} 1 & 0^i \\ v^j & \delta^{ji} \end{pmatrix}, \quad e = \frac{e^i{}_i}{2}$$

Transformation rules follow:

$$\begin{aligned} \rho' &= \rho, \\ p'^i &= p^i + \rho v^i, \\ e' &= e + p^i v_i + \rho \frac{v^2}{2}, \\ j'^i &= j^i + \rho v^i, \end{aligned}$$



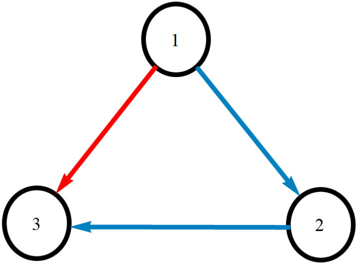
$$\begin{aligned} P'^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\ q'^i &= q^i + e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i) \frac{v^2}{2}. \end{aligned}$$

Galiean transformation of energy

Transitivity :

$$\left.\begin{aligned} e_2 &= e_1 + p_1 v_{12} + \rho\frac{v_{12}^2}{2} \\ e_3 &= e_2 + p_2 v_{23} + \rho\frac{v_{23}^2}{2} \end{aligned}\right\} \rightarrow e_3 = e_1 + p_1 v_{13} + \rho\frac{v_{13}^2}{2}$$

$$p_2 = p_1 + \rho v_{12}, \qquad v_{13} = v_{12} + v_{23}$$



Balance transformations

Absolute

$$\partial_a Z^{abc} = \dot{z}^{bc} + z^{bc}\partial_a u^a + \partial_a z^{\bar{a}bc} = 0$$

Rest frame

$$\begin{aligned}
\dot{\rho} + \partial_i j^i &= 0, \\
\dot{p}^i + \partial_k P^{ik} &= 0^i, \\
\dot{e} + \partial_i q^i &= 0.
\end{aligned}$$

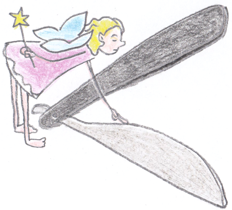
Inertial reference frame

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\dot{p}^i + p^i\partial_k v^k + \partial_k P^{ik} + \rho\dot{v}^i + j^k\partial_k v^i &= 0^i, \\
\dot{e} + e\partial_i v^i + \partial_i q^i + p^i\dot{v}_i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

Interpretations vs. explanations

Quantum physics

- Canonical quantisation. Do we need a Lagrangian?
- Qhydro as transformation or more?
- Stochastic, path integrals, etc...
- Every theory is quantum. Or not. There are some problems. See the discussion of Weinberg (Here is a LINK).



Ockham's razor. *"Pluralitas non est ponenda sine neccesitate"*

"The law that entropy always increases holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations - then so much the worse for Maxwell's equations. If it is found to be contradicted by observation - well, these experimentalists do bungle things sometimes. But if your theory is found to be against the Second Law of Thermodynamics I can give you no hope; there is nothing for it to collapse in deepest humiliation."

Arthur Eddington, New Pathways in Science