Renormalization of composite operators in QCD

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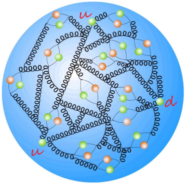
How to gain insight into the structure of hadrons

- Important question: How do hadronic properties emerge from the properties of the constituent partons?
  - For example the proton spin puzzle

    [Aidala et al., 2013],[Leader and Lorcé, 2014],[Deur et al., 2018],[Ji et al., 2021], [Abdulameer et al., 2023]
- Experimentally: Perform high-energy scattering experiments that can resolve the inner hadron structure (e.g. scatter electrons off a proton)
- Hard scale $\Rightarrow$ Factorization between short-range and long-range physics
- Long-range physics described by non-perturbative parton distributions that correspond to hadronic matrix elements of composite operators (PDFs and GPDs)

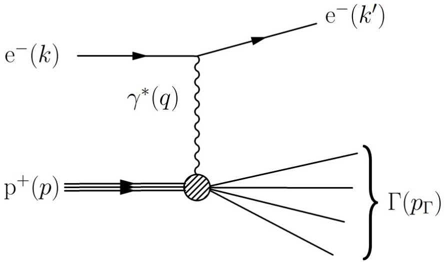


PDFs

PDFs are defined in terms of forward matrix element of composite QCD operators

$$f_i(x) \sim \left\langle p^+(p) \right| \mathcal{O}^i_{\mu_1 \ldots \mu_N} \left| p^+(p) \right\rangle$$

- Probability to find a quark inside the proton with momentum xp $(0 \leq x \leq 1)$
- Encode the longitudinal momentum/polarization carried by partons
- Accessible in inclusive processes like DIS



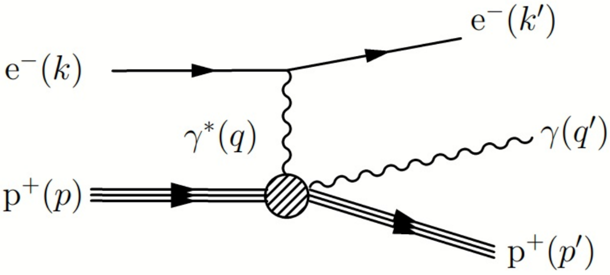
GPDs

- GPDs [Müller et al., 1994a],[Radyushkin, 1996],[Ji, 1997] correspond to non-forward matrix elements of composite operators, $\langle p^+(p)|\, \mathcal{O}^i_{\mu_1\ldots\mu_N}\, |p^+(p')\rangle$, and can be thought of as generalizations of other types of non-perturbative QCD quantities like PDFs, form factors and distribution amplitudes.
- GPDs describe (a) transverse distributions of partons and (b) contributions partonic orbital angular momentum to total hadronic spin

  $\Rightarrow$ Important quantities for describing proton/hadron structure, see e.g. [Pasquini and Boffi, 2008],[Kaiser, 2012],[Bacchetta, 2016]

  $\rightarrow$ Very precise measurements to come in (near) future! (EIC [Boer et al., 2011],[Abdul Khalek et al., 2021]/EicC [Anderle et al., 2021], LHeC [Abelleira Fernandez et al., 2012], JLab22 upgrade [Accardi et al., 2023], ...)
- Accessible in hard exclusive scattering processes
- Theoretically simplest example: deeply-virtual Compton scattering (DVCS)

Deeply-virtual Compton scattering



* Virtuality: $Q^2 = -q^2$
* Bjorken-x: $x_B = \frac{Q^2}{2p \cdot q}$
* Momentum transfer on hadronic target: $t = (p - p')^2 \equiv \delta^2$
* Skewedness: $\xi = \frac{(p-p')^+}{(p+p')^+}$ [lightcone coordinates: $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^3)$]

First experimental measurements of DVCS in the early 2000s (HERMES [Airapetian et al., 2001], CLAS [Stepanyan et al., 2001], H1 [Adloff et al., 2001, Aktas et al., 2005], ZEUS [Chekanov et al., 2003])

Deeply-virtual Compton scattering

In the Bjorken limit ($Q^2 \to \infty$ with x_B, t fixed): Factorization of the DVCS amplitude into non-perturbative GPDs and perturbative coefficient functions

- Coefficient functions correspond to partonic amplitudes
- GPDs correspond to hadronic matrix elements of composite QCD operators

During this talk we will mainly focus on the leading-twist flavor-non-singlet quark operators

$$\mathcal{O} = \mathcal{S}\overline{\psi}\lambda^\alpha \Gamma D_{\mu_2} \dots D_{\mu_N}\psi$$

- Wilson operators (e.g. DVCS):

$$\mathcal{O}_{\mu_1 \dots \mu_N} = \mathcal{S}\overline{\psi}\lambda^\alpha \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N}\psi$$

- Transversity operators (e.g. transverse meson production):

$$\mathcal{O}^T_{\nu\mu_1 \dots \mu_N} = \mathcal{S}\overline{\psi}\lambda^\alpha \sigma_{\nu\mu_1} D_{\mu_2} \dots D_{\mu_N}\psi$$

GPD scale dependence

For phenomenological studies it is important to understand the GPD dependence on the factorization scale μ_F. Typically they are constructed at some reference scale μ^2_{ref} and then evolved to different scales using the GPD evolution equation, which generically takes the following form

[Müller et al., 1994a],[Radyushkin, 1996],[Ji, 1997]

$$\frac{\mathrm{d}\mathcal{G}(x,\xi,t;\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y}\mathcal{P}\left(\frac{x}{y},\frac{\xi}{y}\right)\mathcal{G}(y,\xi,t;\mu^2)$$

This is a generalization of the well-known DGLAP equation

[Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]

$$\frac{\mathrm{d}f(x,\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y}P(y)f\left(\frac{x}{y},\mu^2\right).$$

GPD scale dependence

Because of the direct relation between GPDs and QCD operators, the scale dependence of the distributions is determined by the scale dependence of the operators, characterized by their anomalous dimension

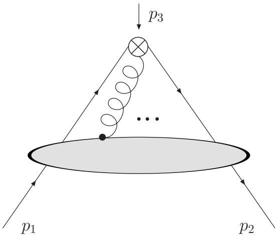
$$\frac{\mathrm{d}[\mathcal{O}]}{\mathrm{d}\ln\mu^2} = \gamma[\mathcal{O}].$$

These anomalous dimensions can be computed perturbatively in QCD by renormalizing the partonic matrix elements of the operators

$$\langle\psi(p_2)|O_{\mu_1\ldots\mu_N}(p_3)|\overline{\psi}(p_1)\rangle.$$

In practical computations, one typically contracts the matrix elements with $\Delta^{\mu_1}\ldots\Delta^{\mu_N}$ where $\Delta^2 = 0$.

Depending on the type of scattering process of interest, one either considers forward ($p_3 = 0$) or non-forward ($p_3 \neq 0$) operator matrix elements. In the latter case one needs to take into account mixing with total-derivative operators.



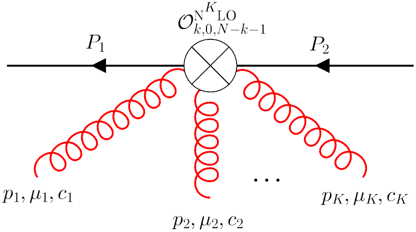
$\rightarrow$  rules!

Operator Feynman rules

To compute the partonic matrix elements, we need to know the Feynman rules for the relevant operator vertices. As their functional form will not depend on their Dirac structure, the rules can be derived generically for operators of the form

$$\mathcal{O}_{k,0,N-k-1} = \mathcal{S}\partial_{\mu_1} \dots \partial_{\mu_k}[\overline{\psi}\Gamma D_{\rho_1} \dots D_{\rho_{N-k-1}}\psi].$$

From the perturbative expansion of the covariant derivatives, it follows that, at the K-loop level, one needs to take into account operator vertices with up to K gluons attached to the operator vertices.



Operator Feynman rules

In principle the operator Feynman rules are derived from the path integral formulation of QCD. However, in practice it turns out that this is not necessary. Instead, one can simply expand the covariant derivatives and replace any field hit by an ordinary derivative by the momentum flowing through this field ($\sim$ Fourier transform to momentum space). In this procedure one has to take care of possible sign conventions:

- Conventions for the sign of the strong coupling in the covariant derivative. One either has

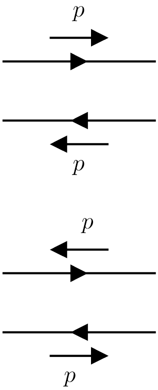
$$D_\mu \psi = \partial_\mu \psi - i g_s A_\mu \psi$$
$$D_\mu \overline{\psi} = \partial_\mu \overline{\psi} + i g_s A_\mu \overline{\psi}$$

or

$$D_\mu \psi = \partial_\mu \psi + i g_s A_\mu \psi$$
$$D_\mu \overline{\psi} = \partial_\mu \overline{\psi} - i g_s A_\mu \overline{\psi}.$$

Operator Feynman rules

- Conventions for the momentum routing:



$$\partial_\mu \psi(p) \to +p_\mu \psi$$

$$\partial_\mu \overline{\psi}(p) \to -p_\mu \overline{\psi}$$

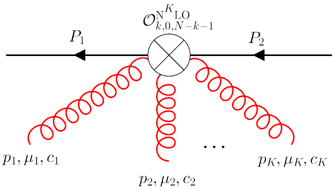
$$\partial_\mu \psi(p) \to -p_\mu \psi$$

$$\partial_\mu \overline{\psi}(p) \to +p_\mu \overline{\psi}$$

Operator Feynman rules

To be generic we use dynamical conventions for easy adaptability to one's own preferences. In particular we set

$$
\begin{aligned}
D_\mu \psi &= \partial_\mu \psi - \delta_D (i g_s A_\mu \psi), \\
D_\mu \overline{\psi} &= \partial_\mu \psi + \delta_D (i g_s A_\mu \overline{\psi}), \\
\partial_\mu \psi(P_2) &\to \delta_2 P_{2\mu}, \\
\partial_\mu \overline{\psi}(P_1) &\to \delta_1 P_{1\mu}
\end{aligned}
$$



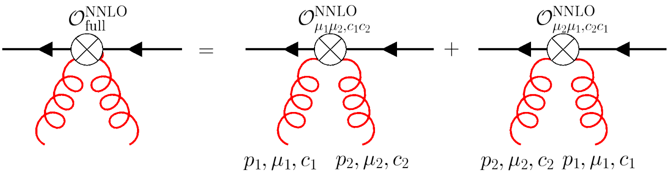
Operator Feynman rules

Some checks of computed rules:

- Total-derivative operators do not contribute in the forward limit. As such, the Feynman rule associated to $\mathcal{O}_{k,0,N-k-1}$ with $k > 0$ is expected to vanish in this limit.
- Finally, several expressions for the operator vertices are already known, especially for $k = 0$. Hence one should also cross-check against such previous results. The Feynman rules for operators with up to four gluons can be found e.g. in [Floratos et al., 1977, Floratos et al., 1979, Mertig and van Neerven, 1996, Kumano and Miyama, 1997, Hayashigaki et al., 1997, Bierenbaum et al., 2009, Klein, 2009, Blümlein, 2001, Velizhanin, 2012a, Velizhanin, 2020, Moch et al., 2017, Moch et al., 2022, Falcioni and Herzog, 2022, Falcioni et al., 2023c, Falcioni et al., 2023b, Falcioni et al., 2023a, Moch et al., 2023, Gehrmann et al., 2023b, Gehrmann et al., 2023c, Kniehl and Velizhanin, 2023] and references therein.

Operator Feynman rules

As an example of the derivation, we consider the NNLO operator vertex



We now need to take into account all possible orderings of derivatives and the 2 gluon fields in $\partial^k[\overline{\psi}\tilde{\Gamma}D^{N-k-1}\psi]$ in which we introduced $\Delta_\mu\partial^\mu \to \partial, \Delta_\mu D^\mu \to D, \Delta_\mu A^\mu \to A, \Delta_\mu\Gamma^\mu \to \tilde{\Gamma}$. Replacing one of the covariant derivatives with a gluon field yields

$$-g_s\delta_D \sum_{i=0}^{N-k-2} \partial^k[\overline{\psi}\tilde{\Gamma}D^i(A\, D^{N-k-2-i}\psi)].$$

Operator Feynman rules

$$-g_s \delta_D \sum_{i=0}^{N-k-2} \partial^k [\overline{\psi} \tilde{\Gamma} D^i (A\, D^{N-k-2-i} \psi)].$$

Next we select the second gluon. Note that we have a choice here: either it comes from the first i covariant derivatives or it comes from the last $N-k-2-i$ ones. Choosing the second gluon to come from the first i covariant derivatives for explicitness, we have

$$g_s^2 \delta_D^2 \sum_{i=0}^{N-k-2} \sum_{j=0}^{i-1} \partial^k [\overline{\psi} \tilde{\Gamma} \partial^j (A_1 \partial^{i-j-1} (A_2 \partial^{N-k-2-i} \psi))].$$

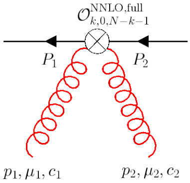
To extract the momentum space Feynman rule from this expression, it is easier to have products of derivatives that only act on one field at the time. For this we can iteratively apply the following relation

$$\partial^k (\phi_1 \partial^m \phi_2) = \sum_{i=0}^{k} \binom{k}{i} (\partial^{k-i} \phi_1)(\partial^{m+i} \phi_2).$$

Operator Feynman rules

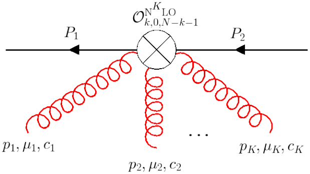
$$g_s^2\delta_D^2 \sum_{i=0}^{N-k-2}\sum_{j=0}^{i-1}\sum_{l=0}^{k}\sum_{m=0}^{j+l}\sum_{n=0}^{i-j+m-1}\binom{k}{l}\binom{j+l}{m}\binom{i-j+m-1}{n}$$
$$\times(\partial^{k-l}\overline{\psi})\tilde{\Gamma}(\partial^{N-k-2-i+n}\psi)(\partial^{j+l-m}A_1)(\partial^{i-j+m-n-1}A_2).$$

Transforming to momentum space and evaluating the sums we then find the following Feynman rule for the NNLO operator vertex



$$g_s^2\delta_D^2 t^{c_1}t^{c_2}(\Delta\cdot\Gamma)\Delta_{\mu_1}\Delta_{\mu_2}\frac{(\Delta\cdot p_1+\Delta\cdot p_2+\delta_1\Delta\cdot P_1+\delta_2\Delta\cdot P_2)^k}{\Delta\cdot p_1\Delta\cdot p_2(\Delta\cdot p_1+\Delta\cdot p_2)(\Delta\cdot p_2+\delta_2\Delta\cdot P_2)}$$
$$\times\Big[\delta_2^{N-k-1}\Delta\cdot p_1\Delta\cdot p_2(\Delta\cdot P_2)^{N-k-1}+\delta_2^{N-k}\Delta\cdot p_1(\Delta\cdot P_2)^{N-k}-(\Delta\cdot p_1+\Delta\cdot p_2)(\Delta\cdot p_2+\delta_2\Delta\cdot P_2)^{N-k}$$
$$+\Delta\cdot p_2(\Delta\cdot p_2+\delta_2\Delta\cdot P_2)(\Delta\cdot p_1+\Delta\cdot p_2+\delta_2\Delta\cdot P_2)^{N-k-1}\Big]$$
$$+[(p_1,\mu_1,c_1)\leftrightarrow(p_2,\mu_2,c_2)]$$

Operator Feynman rules



$$(-g_s\delta_D)^K \left(\prod_{r=1}^{K} t^{c_r}\Delta_{\mu_r}\right)(\Delta\cdot\Gamma)\sum_{l=0}^{k}\binom{k}{l}(\delta_1\Delta\cdot P_1)^{k-l}\prod_{j=0}^{K-1}\sum_{i_{j+1}=0}^{i_j-1}\sum_{m_{j+1}=0}^{i_j-i_{j+1}+m_j-1}\binom{i_j-i_{j+1}+m_j-1}{m_{j+1}}(\Delta\cdot p_{j+1})^{i_j-i_{j+1}+m_j-m_{j+1}-1}(\delta_2\Delta\cdot P_2)^{N-k-i_1+m_K-2}$$

$$+\mathrm{per}_{K-1}[p,\mu,c]$$

- For more details, see [Somogyi and Van Thurenhout, 2024]
- Implementations in both *Mathematica* and *FORM* for the automatic generation of the operator Feynman rules are available at `https://github.com/vtsam/NKLO`
- Cross-checked LO and NLO against private code by J. Gracey for $N = 2, 3$ and for generic k against [Kisselev and Petrov, 2005, Kisselev, 2012].
- Consistent with x-space expressions in [Mikhailov and Volchanskiy, 2020]

Determination of anomalous dimensions from OMEs

The calculations follow a standard workflow (we assume to work in $D = 4 - 2\varepsilon$ dimreg in the $\overline{\text{MS}}$-scheme):

- Generate the Feynman diagrams using *QGRAF* [Nogueira, 1993]
- Feed the output to a *FORM* [Vermaseren, 2000],[Kuipers et al., 2013] program to determine the topologies and color factors
  [Larin et al., 1997],[van Ritbergen et al., 1999],[Herzog et al., 2016]
- Perform the actual diagram calculations with the *FORCER* program [Ruijl et al., 2020], which can efficiently deal with massless propagator-type diagrams in $D = 4 - 2\epsilon$
- Extract the anomalous dimensions from the $1/\epsilon$-pole through renormalization

Operator anomalous dimensions

In forward kinematics, the operators simply renormalize multiplicatively

$$\mathcal{O}_{N+1} = Z_{N,N}[\mathcal{O}_{N+1}].$$

The anomalous dimensions are extracted from the Z-factors as

$$\gamma_{N,N} = -\frac{1}{Z_{N,N}} \frac{\mathrm{d} Z_{N,N}}{\mathrm{d} \ln \mu^2}.$$

In practice, these anomalous dimensions are simply related to the $1/\varepsilon$-pole of the matrix elements. They are related to the splitting functions by a Mellin transformation

$$\gamma_{N,N} = -\int_0^1 \mathrm{d}x \, x^N P_{NS}(x).$$

The latter determine the scale dependence of the PDFs through the DGLAP equation [Gribov and Lipatov, 1972], [Altarelli and Parisi, 1977], [Dokshitzer, 1977]

$$\frac{\mathrm{d} f(x, \mu^2)}{\mathrm{d} \ln \mu^2} = \int_x^1 \frac{\mathrm{d}y}{y} P(y) f\left(\frac{x}{y}, \mu^2\right).$$

Operator anomalous dimensions

These forward anomalous dimensions are known completely up to the 3-loop level, and in certain limits up to the 5-loop level. See Appendix 1 for an extensive literature list.

At the L-loop level, the forward anomalous dimensions in general consist of harmonic sums of maximum weight $w = 2L - 1$ and denominators in $N + \alpha$ (with $\alpha \in \mathbb{N}$) up to the same maximum power.

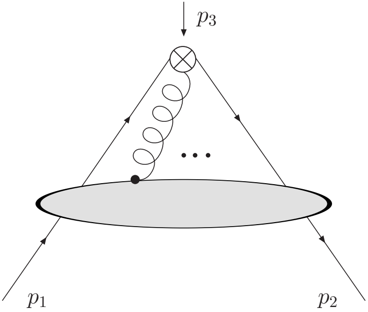
Harmonic sums at argument N are recursively defined by [Vermaseren, 1999, Blümlein and Kurth, 1999]

$$S_{\pm m}(N) = \sum_{i=1}^{N} (\pm 1)^i \, i^{-m}, \quad S_{\pm m_1, m_2, \ldots, m_d}(N) = \sum_{i=1}^{N} (\pm 1)^i \, i^{-m_1} \, S_{m_2, \ldots, m_d}(i) \,.$$

Operator anomalous dimensions

For non-forward kinematics, we need to renormalize matrix elements of the form

$$\langle \psi(p_2) | O_{\mu_1 \ldots \mu_N}(p_3) | \overline{\psi}(p_1) \rangle$$



For simplicity we take $p_1 = p, p_2 = 0$ and $p_3 = -p$. In the leading-twist approximation we consider $\mathcal{O}_N \equiv \Delta^{\mu_1} \ldots \Delta^{\mu_N} O^{NS}_{\mu_1 \ldots \mu_N}$ with $\Delta^2 = 0$.

Operator renormalization: Non-forward kinematics

$$\langle \psi(p_2)|\mathcal{O}(p_3)|\overline{\psi'}(p_1)\rangle$$

In non-forward kinematics ($p_3 \neq 0$), there is mixing with total derivative operators

$$\begin{pmatrix} \mathcal{O}_{N+1} \\ \partial \mathcal{O}_N \\ \vdots \\ \partial^N \mathcal{O}_1 \end{pmatrix} = \begin{pmatrix} Z_{N,N} & Z_{N,N-1} & \dots & Z_{N,0} \\ 0 & Z_{N-1,N-1} & \dots & Z_{N-1,0} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & Z_{0,0} \end{pmatrix} \begin{pmatrix} [\mathcal{O}_{N+1}] \\ [\partial \mathcal{O}_N] \\ \vdots \\ [\partial^N \mathcal{O}_1] \end{pmatrix}$$

Hence we now also have an anomalous dimension matrix (ADM)

$$\hat{\gamma} = -\frac{\mathrm{d} \ln \hat{Z}}{\mathrm{d} \ln \mu^2} = \begin{pmatrix} \gamma_{N,N} & \gamma_{N,N-1} & \cdots & \gamma_{N,0} \\ 0 & \gamma_{N-1,N-1} & \cdots & \gamma_{N-1,0} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \gamma_{0,0} \end{pmatrix}.$$

Diagonal elements: forward anomalous dimensions

Operator renormalization: Non-forward kinematics

The scale dependence of the GPDs is characterized by the evolution equation, which generically takes the following form

[Müller et al., 1994a],[Radyushkin, 1996],[Ji, 1997]

$$\frac{\mathrm{d}\mathcal{G}(x,\xi,t;\mu^2)}{\mathrm{d}\ln\mu^2} = \int_x^1 \frac{\mathrm{d}y}{y}\mathcal{P}\left(\frac{x}{y},\frac{\xi}{y}\right)\mathcal{G}(y,\xi,t;\mu^2)$$

This is a generalization of the DGLAP equation in forward kinematics. The evolution kernel is related to the operator anomalous dimensions as

$$\sum_{k=0}^{N}\gamma_{N,k}y^k = -\int_0^1 \mathrm{d}x\, x^N\,\mathcal{P}(x,y).$$

Because of the mixing, the extraction of the anomalous dimensions from the OMEs is non-trivial. In the following, we discuss 2 possible way of making progress based on (a) a consistency relation for the anomalous dimensions and (b) conformal symmetry arguments.

A consistency relation for the anomalous dimensions

We choose a basis for the operators in which the operators are written as

$$\mathcal{O}^{\mathcal{D}}_{k,N-k} = (\Delta \cdot \partial)^k \{\overline{\psi} \lambda^\alpha (\Delta \cdot \Gamma)(\Delta \cdot D)^{N-k} \psi\}$$

with $\Delta^2 = 0$.

- This choice of operator basis is used for hadronic studies on the lattice, see e.g. [Göckeler et al., 2005] and [Gracey, 2009]
- In this basis, the Wilson anomalous dimensions for low-N operators were computed up to $O(a_s^3)$ (see [Gracey, 2009] for analytical results and [Kniehl and Veretin, 2020] for a numerical extension of these). For the transversity operators, the $O(a_s)$ anomalous dimensions are known [Shifman and Vysotsky, 1981], [Baldracchini et al., 1981], [Artru and Mekhfi, 1990], [Blümlein, 2001]
- We have extended these results by deriving a consistency relation for the anomalous dimensions [Moch and Van Thurenhout, 2021]

A consistency relation for the anomalous dimensions

To set up the consistency relation, we start from a generalization of the $\mathcal{D}$-basis

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} = (\Delta\cdot\partial)^{p}\Big\{(\Delta\cdot D)^{q}\overline{\psi'}\,(\Delta\cdot\Gamma)(\Delta\cdot D)^{r}\psi\Big\}$$

We assume the quark fields to be massless. The operators obey the following relations

- Total derivatives act as

$$\mathcal{O}^{\mathcal{D}}_{p,q,r} \;=\; \mathcal{O}^{\mathcal{D}}_{p-1,q+1,r} + \mathcal{O}^{\mathcal{D}}_{p-1,q,r+1},$$

- Left- and right-derivative operators renormalize with the same renormalization constants

$$\mathcal{O}^{\mathcal{D}}_{k,0,N} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j}\,[\mathcal{O}^{\mathcal{D}}_{k+j,0,N-j}]\,,$$

$$\mathcal{O}^{\mathcal{D}}_{k,N,0} = \sum_{j=0}^{N} Z^{\mathcal{D}}_{N,N-j}\,[\mathcal{O}^{\mathcal{D}}_{k+j,N-j,0}]$$

A consistency relation for the anomalous dimensions

Applying the first relation successively on $\mathcal{O}^{\mathcal{D}}_{N,0,0}$ we derived a recursion-type relation for the bare operators

$$\mathcal{O}^{\mathcal{D}}_{0,N,0} - (-1)^N \sum_{j=0}^{N} (-1)^j \binom{N}{j} \mathcal{O}^{\mathcal{D}}_{j,0,N-j} \;=\; 0.$$

Implementing the renormalization then led to a relation between the renormalization factors $Z^{\mathcal{D}}_{N,k}$ and hence between the anomalous dimensions

$$\boxed{\begin{aligned} \gamma^{\mathcal{D}}_{N,k} \;=& \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,\,j+k} \\ &+ \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j} \end{aligned}}$$

A consistency relation for the anomalous dimensions

$$\boxed{\begin{aligned}\gamma^{\mathcal{D}}_{N,k} &= \binom{N}{k}\sum_{j=0}^{N-k}(-1)^j\binom{N-k}{j}\gamma_{j+k,\,j+k} \\ &\quad + \sum_{j=k}^{N}(-1)^k\binom{j}{k}\sum_{l=j+1}^{N}(-1)^l\binom{N}{l}\gamma^{\mathcal{D}}_{l,\,j}\end{aligned}}$$

- ✓ Order-independent consistency check
- ✓ Can be used to construct the full ADM from the knowledge of the forward anomalous dimensions $\gamma_{N,N}$ + boundary condition to ensure uniqueness of the solution ($\gamma^{\mathcal{D}}_{N,0}$, from Feynman diagrams)

4-step algorithm for constructing the ADM

1. Calculate

$$\binom{N}{k}\sum_{j=0}^{N-k}(-1)^j\binom{N-k}{j}\gamma_{j+k,\,j+k}$$

   and construct an Ansatz for the off-diagonal piece

2. Calculate

$$\sum_{j=k}^{N}(-1)^k\binom{j}{k}\sum_{l=j+1}^{N}(-1)^l\binom{N}{l}\gamma^{\mathcal{D}}_{l,j}$$

3. Substitute into the consistency relation $\Rightarrow$ System of equations, solution not necessarily unique $\Rightarrow$ Need boundary condition!
4. Determine all-N expression for $\gamma^{\mathcal{D}}_{N,0}$ from Feynman diagrams

Feynman diagrams and $\gamma^{\mathcal{D}}_{N,0}$

From the general consistency relation for $k = 0$ we can derive a recursion-type relation for the last column

$$\gamma^{\mathcal{D}}_{N,0} = (-)^N \left[\sum_{i=0}^{N} \gamma^{\mathcal{D}}_{N,i} - \sum_{j=1}^{N-1} (-)^j \binom{N}{j} \gamma^{\mathcal{D}}_{j,0} \right]$$

$$\begin{pmatrix} \gamma_{N,N} & \gamma^{\mathcal{D}}_{N,N-1} & \gamma^{\mathcal{D}}_{N,N-2} & \cdots & \gamma^{\mathcal{D}}_{N,k} & \cdots & \gamma^{\mathcal{D}}_{N,0} \\ 0 & \gamma_{N-1,N-1} & \gamma^{\mathcal{D}}_{N-1,N-2} & \cdots & \gamma^{\mathcal{D}}_{N-1,k} & \cdots & \gamma^{\mathcal{D}}_{N-1,0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \gamma_{k,k} & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & \gamma_{0,0} \end{pmatrix}$$

Feynman diagrams and $\gamma^{\mathcal{D}}_{N,0}$

Remember now the renormalization pattern for right-derivative operators

$$\mathcal{O}_{N+1} = Z_{N,N}[\mathcal{O}_{N+1}] + Z_{N,N-1}[\partial\mathcal{O}_N] + \cdots + Z_{N,0}[\partial^N\mathcal{O}_1].$$

From this, it is clear that there is a direct relation between the bare matrix elements of the operators $\mathcal{O}_{N+1}$ and the sum of anomalous dimensions

$$\mathcal{B}(N+1) \equiv \mathcal{O}_{N+1}/\epsilon \sim \sum_{i=0}^{N} \gamma^{\mathcal{D}}_{N,i}.$$

Hence, it follows that there is a direct relation between the bare OMEs and the last column of the mixing matrix

$$\gamma^{\mathcal{D}}_{N,0} = \sum_{i=0}^{N}(-1)^i\binom{N}{i}\mathcal{B}(i+1).$$

The OMEs can in general depend on the gauge parameter. Using a general covariant gauge, the independence of the anomalous dimensions of the gauge parameter then provides a check on our diagram calculations.

Structure of the anomalous dimensions and sums

As we saw before, the forward (diagonal) anomalous dimensions consist of harmonic sums and denominators. We expect the off-diagonal elements of the ADM to have a similar structure. So, to apply our algorithm, we need to calculate single and double sums over such structures

$$\gamma^{\mathcal{D}}_{N,k} = \binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma_{j+k,j+k} + \sum_{j=k}^{N} (-1)^k \binom{j}{k} \sum_{l=j+1}^{N} (-1)^l \binom{N}{l} \gamma^{\mathcal{D}}_{l,j}.$$

Such sums can be dealt with using principles of symbolic summation, see e.g. [Graham et al., 1989, Kauers and Paule, 2011] for extensive overviews. In particular, the *MATHEMATICA* package *SIGMA* [Schneider, 2007, Schneider, 2013], which uses creative telescoping to solve summation problems, is very helpful.

Structure of the anomalous dimensions and sums

Telescoping: Suppose we have a summation $\sum_{k=a}^{N} f(k)$

$\rightarrow$ Find function $g(N)$ such that the summand can be written as

$$f(k) = g(k+1) - g(k)$$

$$\begin{aligned} \Rightarrow \sum_{k=a}^{N} f(k) &= \sum_{k=a}^{N} g(k+1) - \sum_{k=a}^{N} g(k) \\ &= g(N+1) - g(a) \end{aligned}$$

Structure of the anomalous dimensions and sums

Creative telescoping [Zeilberger, 1991]: Suppose we have the summation

$$\sum_{k=a}^{b} f(n,k) \equiv S(n)$$

$\rightarrow$ Find functions $c_0(n), ..., c_d(n)$ and $g(n,k)$ such that

$$g(n,k+1) - g(n,k) = c_0(n)f(n,k) + ... + c_d(n)f(n+d,k)$$

Now apply summation on both sides of the equation

$$\Rightarrow g(n,b+1) - g(n,a) = c_0(n)\sum_{k=a}^{b} f(n,k) + ... + c_d(n)\sum_{k=a}^{b} f(n+d,k)$$

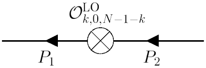
$\Rightarrow$ Inhomogeneous recurrence for original sum

$$q(n) = c_0(n)S(n) + ... + c_d(n)S(n+d)$$

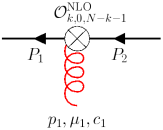
Calculating sums

- In this way, *SIGMA* generates and solves recurrence for given summation problem
- Solution consists of solution set for homogeneous recurrence + particular solution
- For final closed expression of summation: Determine linear combination of solutions that has same initial values as the given sum

Let us now, as an example, look at the computation of the 1-loop ADM



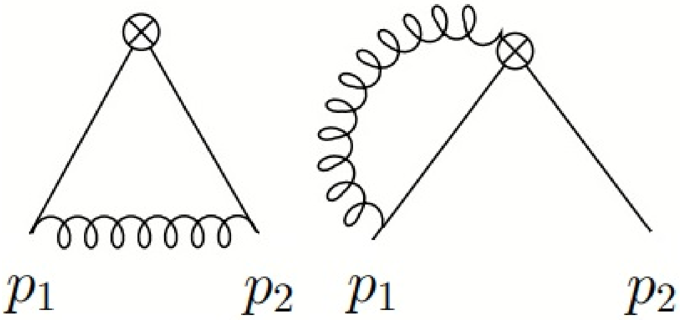
$$(\Delta\cdot\Gamma)\delta_2^{N-k-1}(\Delta\cdot P_2)^{N-k-1}(\delta_1\Delta\cdot P_1+\delta_2\Delta\cdot P_2)^k$$



$$g_s\, t^{c_1}(\Delta\cdot\Gamma)\Delta_{\mu_1}\delta_D\,\frac{(\delta_1\Delta\cdot P_1+\delta_2\Delta\cdot P_2+\Delta\cdot p_1)^k}{\Delta\cdot p_1}\left[(\delta_2\Delta\cdot P_2)^{N-k-1}-(\Delta\cdot p_1+\delta_2\Delta\cdot P_2)^{N-k-1}\right]$$

Example: 1-loop ADM

At the 1-loop level, there are only 2 Feynman diagrams



Example: 1-loop ADM

The calculation of the bare OMEs is then straightforward and we find

$$\mathcal{O}^{(0)}_{N+1} = 1 + \frac{a_s}{\epsilon} C_F \Big(2S_1(N+1) - \frac{2}{N+2} - 1 \Big).$$

The last column of the mixing matrix can be calculated as

$$\gamma_{N,0} = \sum_{i=0}^{N} (-1)^i \binom{N}{i} \mathcal{B}(i+1)$$

which in this example implies

$$\gamma^{(0)}_{N,0} = C_F \sum_{i=0}^{N} (-1)^i \binom{N}{i} \left(2S_1(i+1) - \frac{2}{i+2} - 1 \right).$$

Example: 1-loop ADM

Using [Vermaseren, 1999]

$$\begin{aligned}
\sum_{j=0}^{N'} (-1)^j \binom{N'}{j} &= \delta_{N',0} \\
\sum_{j=0}^{N'} (-1)^j \binom{N'}{j} \frac{1}{m+j} &= \frac{1}{\binom{N'+m}{N'}} \frac{1}{m} \\
\sum_{j=0}^{N'} (-1)^j \binom{N'}{j} S_1(m+j) &= \frac{-1}{\binom{N'+m}{N'}} \frac{1}{N'},
\end{aligned}$$

we find

$$\gamma_{N,0}^{(0)} = C_F \left(\frac{2}{N+2} - \frac{2}{N} \right).$$

Example: 1-loop ADM

Next we need the single sum of the forward anomalous dimensions

$$\binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma^{(0)}_{j+k,j+k}.$$

Using *SIGMA* we find

$$\binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} \gamma^{(0)}_{j+k,j+k} = C_F \left(\frac{-2(k+1)}{N+2} + \frac{2(k+2)}{N+1} - \frac{4}{N-k} \right).$$

Based on this, we then choose the following Ansatz for the off-diagonal piece

$$\gamma^{(0)}_{N,k} = C_F \left(\frac{a_1}{N+2} + \frac{a_2}{N+1} + \frac{a_3}{N-k} \right).$$

Example: 1-loop ADM

Again using *SIGMA* for the double sum of the Ansatz

$$\frac{1}{C_F}\sum_{j=k}^{N}(-1)^k\binom{j}{k}\sum_{l=j+1}^{N}(-1)^l\binom{N}{l}\gamma_{l,j}^{(0)} =$$
$$a_1\left[\frac{-2-k}{N+1}+\frac{2+k}{N+2}\right]-\frac{a_2}{N+1}-\frac{a_3}{N-k}.$$

We now substitute the results into the consistency relation

$$\left(\frac{a_1}{N+2}+\frac{a_2}{N+1}+\frac{a_3}{N-k}\right)=\left(\frac{-2(k+1)}{N+2}+\frac{2(k+2)}{N+1}-\frac{4}{N-k}\right)$$
$$+a_1\left[\frac{-2-k}{N+1}+\frac{2+k}{N+2}\right]-\frac{a_2}{N+1}-\frac{a_3}{N-k}$$

$$\Rightarrow \gamma_{N,k}^{(0)} = C_F\left(\frac{2}{N+2}-\frac{2}{N-k}\right).$$

Results and discussion

Application of this method allowed us to extend the low-N results in the following ways

- Large n_f: 5-loop Wilson, 4-loop transversity anomalous dimensions [Moch and Van Thurenhout, 2021, Van Thurenhout, 2022] (see also [Van Thurenhout and Moch, 2022] for all-order results in this limit)
- Large n_c: 2-loop Wilson anomalous dimensions [Moch and Van Thurenhout, 2021] (subleading color analysis in progress)

Main advantage: The full procedure can be automated using computer algebra methods, e.g.

- Diagram computations using e.g. *FORCER* [Ruijl et al., 2020] in *FORM*
- Evaluation of sums using e.g. the *Mathematica* package *SIGMA* [Schneider, 2007, Schneider, 2013]

$\Rightarrow$ In principle straightforward to go to higher orders in perturbation theory!

Anomalous dimensions from conformal symmetry

Anomalous dimensions from conformal symmetry

- Instead of working with physical 4D QCD, one considers QCD in $D = 4 - 2\varepsilon$ dimensions at the critical point
- The anomalous dimensions $\gamma^{\mathcal{C}}$ can then be reconstructed using consistency relations coming from the conformal algebra
- The physical kernels have the same functional form as the critical ones, up to terms associated to the breaking of conformal symmetry: QCD beta-function and the conformal anomaly (currently known to two-loop accuracy [Müller, 1991, Braun et al., 2016, Braun et al., 2017])

$$\gamma_{N,k}^{\mathcal{C},(1)} = -\frac{\gamma_{N,N}^{(0)} - \gamma_{k,k}^{(0)}}{a(N,k)} \left\{ -2(2k+3)\left(\beta_0 + \gamma_{k,k}^{(0)}\right)\vartheta_{N,k} + 2\Delta_{N,k}^{\mathcal{C},(0)} \right\}$$

- As generically the L-loop anomalous dimensions depend only on the $(L-1)$-loop conformal anomaly [Müller, 1991], they could be calculated up to three loops using this approach [Braun et al., 2017]

Connecting results

The 2 approaches above follow independent methods and use different bases for the total-derivative operators. They can be connected to each other by constructing a similarity transformation between the 2 bases

[Van Thurenhout, 2024]

$$\gamma^{\mathcal{D}}_{N,k} = \frac{(-1)^k (N+1)!}{(k+1)!} \sum_{l=k}^{N} (-1)^l \binom{N}{l} \frac{l!\,(3+2l)}{(N+l+3)!} \sum_{j=k}^{l} \binom{j}{k} \frac{(j+k+2)!}{j!} \gamma^{\mathcal{C}}_{l,j}$$

$$\gamma^{\mathcal{C}}_{N,k} = (-1)^k \frac{k!}{N!} (3+2k) \sum_{l=k}^{N} (-1)^l \binom{N}{l} \frac{(N+l+2)!}{(l+1)!} \sum_{j=k}^{l} \binom{j}{k} \frac{(j+1)!}{(j+k+3)!} \gamma^{\mathcal{D}}_{l,j}$$

- ✓ Cross-check independent computations
- ✓ Learn about functional form of the ADM

Neat consequence: Validation of conformal anomaly computations

In conformal schemes, the conformal anomaly constitutes vital input for the computation of the off-diagonal elements of the ADM. At the 1-loop level it is given by [Braun et al., 2017, Braun et al., 2016, Braun and Manashov, 2014]

$$\Delta_{N,k}^{\mathcal{C},(0)} = 2C_F(2k+3)a(N,k)\left(\frac{A_{N,k} - S_1(N+1)}{(k+1)(k+2)} + \frac{2A_{N,k}}{a(N,k)}\right)\vartheta_{N,k}$$

with

$$A_{N,k} = S_1\left(\frac{N+k+2}{2}\right) - S_1\left(\frac{N-k-2}{2}\right) + 2S_1(N-k-1) - S_1(N+1).$$

The two-loop conformal anomaly is also known, although no closed-form expression exists thus far [Braun et al., 2017].

Validation of conformal anomaly computations

Using our new similarity transformation, the conformal anomaly (in the $\mathcal{C}$-basis) can be written in terms of the leading-color anomalous dimensions in the $\mathcal{D}$-basis at 1 order in a_s higher. For example the 1-loop conformal anomaly can be written as

$$\Delta^{\mathcal{C},(0)}_{N,k} = (2k+3)\left(\gamma^{(0)}_{k,k} + \frac{22}{3}C_F\right) - \frac{(N-k)(N+k+3)}{2(\gamma^{(0)}_{N,N} - \gamma^{(0)}_{k,k})} R_{N,l}\tilde{R}_{j,k}\gamma^{\mathcal{D},(1)}_{l,j}\Big|_{\mathrm{LC}}$$

Using the expression for $\gamma^{\mathcal{D},(1)}_{N,k}$ computed in [Moch and Van Thurenhout, 2021] we find exact agreement for $\Delta^{\mathcal{C},(0)}_{N,k}$.

- Similar expressions at higher orders
- Leading-color anomalous dimensions in $\mathcal{D}$-basis in principle straightforward to compute
- Representation of the anomaly in terms of sums, which can be efficiently evaluated for fixed (N, k) using e.g. the *SUMMER* package [Vermaseren, 1999] in *FORM* [Vermaseren, 2000, Kuipers et al., 2013]

Summary and outlook

We can gain insight into proton structure by scattering elementary particles off protons. The relevant information is then typically represented by parton distributions

- Inclusive processes (forward kinematics): PDFs
- Exclusive processes (non-forward kinematics): GPDs.

These distributions are defined in terms of hadronic matrix elements of QCD operators and hence they are non-perturbative. However, their scale dependence is determined by the anomalous dimensions of the corresponding operators, which can be calculated order per order in perturbation theory.

The calculation of these anomalous dimensions is in principle straightforward in the forward case. For non-forward kinematics life becomes more complicated because of mixing with total derivative operators.

Summary and outlook

- We have discussed two methods to reconstruct the anomalous dimensions using (a) a consistency relation and (b) conformal symmetry arguments
- These a priori different methods can be compared by applying a basis transformation
- Generalize method to flavor singlet operators $\rightarrow$ more complicated operator mixing:
  * Gluon operators (generic Feynman rules in progress)
  * Alien operators

End[1]

Thank you for your attention!

[1] Part of this work has been supported by grant K143451 of the National Research, Development and Innovation Fund in Hungary.

Appendices and references

Computations of evolution kernels and anomalous dimensions

Forward evolution kernels/anomalous dimensions:

[Gross and Wilczek, 1973, Gross and Wilczek, 1974, Floratos et al., 1977, Gonzalez-Arroyo et al., 1979, Floratos et al., 1979, Gonzalez-Arroyo and Lopez, 1980, Gonzalez-Arroyo et al., 1980, Curci et al., 1980, Furmanski and Petronzio, 1980, Shifman and Vysotsky, 1981, Baldracchini et al., 1981, Artru and Mekhfi, 1990, Gracey, 1994, Gracey, 1996, Hayashigaki et al., 1997, Kumano and Miyama, 1997, Blumlein et al., 1997b, Larin et al., 1997, Vogelsang, 1998, Bennett and Gracey, 1998, Blumlein and Vogt, 1998, Blumlein et al., 1998a, Blumlein et al., 1998b, van Neerven and Vogt, 2000, Blümlein, 2001, Gracey, 2003a, Gracey, 2003b, Vogt et al., 2004, Moch et al., 2004, Blumlein, 2004, Gracey, 2006a, Gracey, 2006b, Blumlein et al., 2009, Bierenbaum et al., 2009, Vogt et al., 2010a, Soar et al., 2010, Vogt et al., 2010b, Ablinger et al., 2011, Velizhanin, 2012a, Velizhanin, 2012b, Ablinger et al., 2014a, Ablinger et al., 2014b, Moch et al., 2014, Ruijl et al., 2016, Davies et al., 2017, Moch et al., 2017, Ablinger et al., 2017, Vogt et al., 2018, Moch et al., 2018, Behring et al., 2019, Herzog et al., 2019, Velizhanin, 2020, Blümlein et al., 2021, Blümlein et al., 2022b, Moch et al., 2022, Blümlein et al., 2022a, Falcioni and Herzog, 2022, Blümlein, 2023, Gehrmann et al., 2023c, Gehrmann et al., 2023a, Gehrmann et al., 2023b, Falcioni et al., 2023c, Ji et al., 2023, Falcioni et al., 2023b, Falcioni et al., 2023a, Moch et al., 2023]

Computations of evolution kernels and anomalous dimensions

Non-forward evolution kernels/anomalous dimensions:

[Efremov and Radyushkin, 1980, Makeenko, 1981, Shifman and Vysotsky, 1981, Baldracchini et al., 1981, Geyer, 1982, Gribov et al., 1983, Geyer et al., 1985, Braunschweig et al., 1986, Dittes et al., 1988, Balitsky and Braun, 1989, Artru and Mekhfi, 1990, Müller, 1991, Müller, 1994, Müller et al., 1994b, Ji, 1997, Radyushkin, 1997, Balitsky and Radyushkin, 1997, Blumlein et al., 1997a, Martin and Ryskin, 1998, Belitsky and Müller, 1998, Hoodbhoy and Ji, 1998, Belitsky and Müller, 1999a, Radyushkin, 1999, Blümlein et al., 1999, Belitsky et al., 1999, Belitsky and Müller, 1999b, Belitsky et al., 2000a, Belitsky et al., 2000b, Belitsky and Müller, 2000, Blümlein, 2001, Mikhailov and Vladimirov, 2009a, Mikhailov and Vladimirov, 2009b, Gracey, 2009, Gracey, 2011a, Gracey, 2011b, Braun and Manashov, 2013, Braun and Manashov, 2014, Manashov and Strohmaier, 2015, Braun et al., 2017, Braun et al., 2019a, Braun et al., 2019b, Kniehl and Veretin, 2020, Braun et al., 2021, Moch and Van Thurenhout, 2021, Braun et al., 2022, Bertone et al., 2022, Van Thurenhout, 2022, Van Thurenhout and Moch, 2022, Ji et al., 2023, Van Thurenhout, 2024, Bertone et al., 2023]

Some comments on *FORCER*

- *FORM* [Vermaseren, 2000], [Kuipers et al., 2013] program for the reduction of four-loop massless propagator-type integrals to master integrals
- Parametric IBP reductions
- Often possible to avoid explicit IBP reductions by reducing topologies to simpler ones (1-loop integrals, triangle rule, ...) $\rightarrow$ Automatized!
- Less diagrams for which actual IBP reductions are necessary, special rules for these

More details can be found in the original paper [Ruijl et al., 2020].

GPDs

$$F^q \equiv \int \frac{dz^-}{2\pi} e^{ix\chi^+ z^-} \langle p' | \overline{\psi}(-z/2)\gamma^+\psi(z/2) | p \rangle \sim H(x,\chi,t)\overline{\psi}(p')\gamma^+\psi(p)$$
$$+ E(x,\chi,t)\overline{\psi}(p')\frac{i\sigma^{+\nu}\tilde{\Delta}_\nu}{2m_p}\psi(p)$$
$$+ \text{higher twist}$$

$$\int dx\, x^N F^q \sim \overline{\psi}(0)\gamma^+ D^N \psi(0)$$

see e.g. [Diehl, 2003]. Here χ is the skewedness

$$\chi = \frac{p^+ - p'^+}{p^+ + p'^+}.$$

For some four-vector $v \equiv (v^0, v^1, v^2, v^3)$ light-cone coordinates are defined as

$$v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^3), \quad \vec{v} = (v^1, v^2).$$

Five-loop ADM in the leading-n_f limit

- By analyzing the leading-n_f anomalous dimensions up to 4 loops, it becomes clear that certain patterns start to emerge.
- The majority of terms in the L-loop anomalous dimensions can be deduced from the expression of the $(L-1)$-loop ones.
- What is left then is a small number of unknown terms, which can be fixed by using our consistency relation.
- This is how we determined the expression for the 5-loop Wilson anomalous dimensions. Note that this in principle can be extended to arbitrary orders.
- This method is also used to determine the leading-n_f mixing matrices for the transversity operators.

All-order results in the large-n_f limit

In [Gracey, 1994] and [Gracey, 2003b] the all-order expressions for the Wilson and transversity forward anomalous dimensions in the leading-n_f approximation were computed[2].

The calculation relied on exact conformal symmetry at the Wilson-Fisher critical point [Braun et al., 2019c], in which case propagators in the model simply have a power law structure. The anomalous dimensions calculated this way are then functions of the spacetime dimension D and n_f.

In [Van Thurenhout and Moch, 2022] we extended this programme to the computation of the off-diagonal elements of the ADM.

[2] An independent computation in x-space, based on summation of renormalon-chain insertions, was performed in [Mikhailov, 1998b],[Mikhailov, 1998a],[Mikhailov, 2000].

All-order results in the large-n_f limit

The general expression from which the anomalous dimensions can be extracted is[3]

$$\gamma\mathcal{O}(z_1,z_2) = \frac{\mu(\mu-1)}{2(\mu-2)(2\mu-1)}\eta\Bigg\{\int_0^1 \mathrm{d}\alpha\,\frac{\overline{\alpha}^{\mu-1}}{\alpha}\left(2[\mathcal{O}(z_1,z_2)] - [\mathcal{O}(z_{12}^{\alpha},z_2)] - [\mathcal{O}(z_1,z_{21}^{\alpha})]\right)$$
$$- (\mu-\delta)^2\int_0^1 \mathrm{d}\alpha\int_0^{\overline{\alpha}} \mathrm{d}\beta\,(1-\alpha-\beta)^{\mu-2}[\mathcal{O}(z_{12}^{\alpha},z_{21}^{\beta})] + \frac{\mu-1}{\mu}[O(z_1,z_2)]\Bigg\}$$

with

$$\mu = \frac{D}{2} = 2-\varepsilon_* = 2 + a_s\beta_0\Big|_{n_f} = 2 - \frac{2}{3}n_f a_s$$
$$\eta = \frac{1}{n_f}\frac{(\mu-2)(2\mu-1)\Gamma(2\mu)}{\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)}$$
$$z_{12}^{\alpha} = z_1\overline{\alpha} + z_2\alpha,\ \ \overline{\alpha} = 1-\alpha$$

δ is a parameter controlling the Dirac structure of the considered operators (1 for Wilson and 2 for transversity).

[3] We thank A. Manashov for useful discussions on this subject.

All-order results in the large-n_f limit

Depending on the case of interest, we now substitute different expressions for the moments of the non-local operators $\mathcal{O}(z_1, z_2)$

- Forward kinematics

$$\mathcal{O}(z_1, z_2) \to z_{12}^{N-1} = (z_1 - z_2)^{N-1}$$

  The forward anomalous dimensions then simply correspond to the prefactor of $(z_1 - z_2)^{N-1}$ and agree with [Gracey, 1994],[Gracey, 2003b]

- Non-forward kinematics: Use that the non-local operators act as generating functions for local ones [Braun et al., 2017]

$$\begin{aligned}
[\mathcal{O}(z_1, z_2)] &= \sum_{m,k} \frac{z_1^m z_2^k}{m!\ k!} [\bar{\psi}(x)(\overleftarrow{D} \cdot \Delta)^k (\Delta \cdot \Gamma)(\Delta \cdot \overrightarrow{D})^m \psi(x)] \\
&= \sum_{m,k} \frac{z_1^m z_2^k}{m!\ k!} [\mathcal{O}_{0,k,m}]
\end{aligned}$$

The latter operators can be written in terms of operators without covariant derivatives acting on ψ as

$$\mathcal{O}_{0,N-k,k} = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathcal{O}_{j,N-j,0}.$$

It then follows that

$$[\mathcal{O}(z_1, z_2)] = \sum_{k=0}^{N} \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} \frac{z_1^{N-k} z_2^k}{k!\,(N-k)!} [\mathcal{O}_{j,N-j}].$$

All-order results in the large-n_f limit

The resulting integrals can be computed for fixed values of N. Taking the N-th derivative with respect to z_1 and take $z_1, z_2 \to 0$, the expression takes the form

$$\gamma\mathcal{O}(z_1, z_2) = \gamma_{N,N}[\mathcal{O}_{0,N}] + \gamma_{N,N-1}[\mathcal{O}_{1,N-1}] + \gamma_{N,N-2}[\mathcal{O}_{2,N-2}] + \cdots + \gamma_{N,0}[\mathcal{O}_{N,0}],$$

from which the all-order expressions for $\gamma_{N,k}$ with $k = 0, 1, \ldots, N$ can be read off. The results agree with what was computed in

[Moch and Van Thurenhout, 2021], [Van Thurenhout, 2022]

All-order results in the large-n_f limit

Non-trivial example:

$$\gamma_{3,2} = -4(a_s n_f - 3)[36 + a_s n_f(2a_s n_f - 15)]\mathcal{F}(a_s, n_f),$$
$$\gamma_{3,1} = 9[18 + a_s n_f(2a_s n_f - 11)]\mathcal{F}(a_s, n_f),$$
$$\gamma_{3,0} = -24(a_s n_f - 3)\mathcal{F}(a_s, n_f)$$

for the Wilson operators and

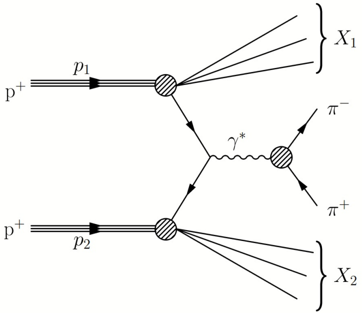
$$\gamma^T_{3,2} = (3 - a_s n_f)[135 + 8a_s n_f(a_s n_f - 6)]\mathcal{F}(a_s, n_f),$$
$$\gamma^T_{3,1} = 9[15 + a_s n_f(2a_s n_f - 7)]\mathcal{F}(a_s, n_f),$$
$$\gamma^T_{3,0} = \frac{-3}{2a_s n_f - 3}[45 + 4a_s n_f(4a_s n_f - 9)]\mathcal{F}(a_s, n_f)$$

for the transversity ones with

$$\mathcal{F}(a_s, n_f) = -\frac{2^{3-4a_s n_f/3}}{9\pi^{3/2} n_f} \frac{\Gamma(5/2 - 2a_s n_f/3)\sin(2\pi a_s n_f/3)}{\Gamma(6 - 2a_s n_f/3)}.$$

Example of an inclusive process

- Inclusive polarized Drell-Yan



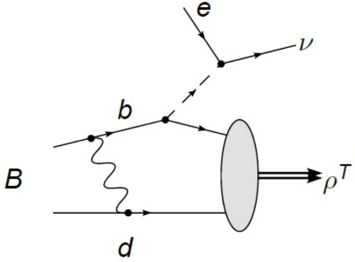
Distributions: Transversity distributions (TDFs) $h^T(x, \mu_f^2)$

[Ralston and Soper, 1979], [Artru and Mekhfi, 1990], [Jaffe and Ji, 1991], [Jaffe and Ji, 1992], [Cortes et al., 1992]

- ◇ Difference in probabilities of finding a parton in a transversly polarized nucleon polarized parallel to the nucleon spin and an oppositely polarized one
- ◇ Studied e.g. by the STAR experiment at RHIC [Adamczyk et al., 2015]

Example of an exclusive process

- Exclusive production of transversely polarized ρ-meson



Distributions: Transverse distribution amplitudes (DAs) $\phi(x, \mu_F^2)$

[Lepage and Brodsky, 1980]

- ◇ Measure parton distributions within mesons
- ◇ Important input for e.g. LHCb [Aaij et al., 2013]

References I

[Aaij et al., 2013] Aaij, R. et al. (2013).
Implications of LHCb measurements and future prospects.
Eur. Phys. J. C, 73(4):2373.

[Abdul Khalek et al., 2021] Abdul Khalek, R. et al. (2021).
Science Requirements and Detector Concepts for the Electron-Ion Collider: EIC Yellow Report.

[Abdulameer et al., 2023] Abdulameer, N. J. et al. (2023).
Measurement of direct-photon cross section and double-helicity asymmetry at $s = 510$ gev in $p \to + p \to$ collisions.
Physical Review Letters, 130(25).

[Abelleira Fernandez et al., 2012] Abelleira Fernandez, J. L. et al. (2012).
A Large Hadron Electron Collider at CERN: Report on the Physics and Design Concepts for Machine and Detector.
J. Phys. G, 39:075001.

References II

[Ablinger et al., 2014a] Ablinger, J., Behring, A., Blümlein, J., De Freitas, A., Hasselhuhn, A., von Manteuffel, A., Round, M., Schneider, C., and Wißbrock, F. (2014a).
The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity.
Nucl. Phys. B, 886:733–823.

[Ablinger et al., 2014b] Ablinger, J., Behring, A., Blümlein, J., De Freitas, A., von Manteuffel, A., and Schneider, C. (2014b).
The 3-loop pure singlet heavy flavor contributions to the structure function $F_2(x, Q^2)$ and the anomalous dimension.
Nucl. Phys. B, 890:48–151.

References III

[Ablinger et al., 2017] Ablinger, J., Behring, A., Blümlein, J., De Freitas, A., von Manteuffel, A., and Schneider, C. (2017).
The three-loop splitting functions $P_{qg}^{(2)}$ and $P_{gg}^{(2,N_F)}$.
Nucl. Phys. B, 922:1–40.

[Ablinger et al., 2011] Ablinger, J., Blümlein, J., Klein, S., Schneider, C., and Wissbrock, F. (2011).
The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity.
Nucl. Phys. B, 844:26–54.

[Accardi et al., 2023] Accardi, A. et al. (2023).
Strong Interaction Physics at the Luminosity Frontier with 22 GeV Electrons at Jefferson Lab.

References IV

[Adamczyk et al., 2015] Adamczyk, L. et al. (2015).
Observation of Transverse Spin-Dependent Azimuthal Correlations of Charged Pion Pairs in $p^{\uparrow} + p$ at $\sqrt{s} = 200$ GeV.
Phys. Rev. Lett., 115:242501.

[Adloff et al., 2001] Adloff, C. et al. (2001).
Measurement of deeply virtual Compton scattering at HERA.
Phys. Lett. B, 517:47–58.

[Aidala et al., 2013] Aidala, C. A., Bass, S. D., Hasch, D., and Mallot, G. K. (2013).
The Spin Structure of the Nucleon.
Rev. Mod. Phys., 85:655–691.

References V

[Airapetian et al., 2001] Airapetian, A. et al. (2001).
Measurement of the beam spin azimuthal asymmetry associated with deeply virtual Compton scattering.
Phys. Rev. Lett., 87:182001.

[Aktas et al., 2005] Aktas, A. et al. (2005).
Measurement of deeply virtual compton scattering at HERA.
Eur. Phys. J. C, 44:1–11.

[Altarelli and Parisi, 1977] Altarelli, G. and Parisi, G. (1977).
Asymptotic Freedom in Parton Language.
Nucl. Phys. B, 126:298–318.

[Anderle et al., 2021] Anderle, D. P. et al. (2021).
Electron-ion collider in China.
Front. Phys. (Beijing), 16(6):64701.

References VI

[Artru and Mekhfi, 1990] Artru, X. and Mekhfi, M. (1990).
Transversely Polarized Parton Densities, their Evolution and their Measurement.
Z. Phys. C, 45:669.

[Bacchetta, 2016] Bacchetta, A. (2016).
Where do we stand with a 3-D picture of the proton?
Eur. Phys. J. A, 52(6):163.

[Baldracchini et al., 1981] Baldracchini, F., Craigie, N. S., Roberto, V., and Socolovsky, M. (1981).
A Survey of Polarization Asymmetries Predicted by QCD.
Fortschritte der Physik, 29(11-12):505–550.

[Balitsky and Braun, 1989] Balitsky, I. I. and Braun, V. M. (1989).
Evolution Equations for QCD String Operators.
Nucl. Phys. B, 311:541–584.

References VII

[Balitsky and Radyushkin, 1997] Balitsky, I. I. and Radyushkin, A. V. (1997).
Light ray evolution equations and leading twist parton helicity dependent nonforward distributions.
Phys. Lett. B, 413:114–121.

[Behring et al., 2019] Behring, A., Blümlein, J., De Freitas, A., Goedicke, A., Klein, S., von Manteuffel, A., Schneider, C., and Schönwald, K. (2019).
The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements.
Nucl. Phys. B, 948:114753.

References VIII

[Belitsky et al., 2000a] Belitsky, A. V., Freund, A., and Müller, D. (2000a).

Evolution kernels of skewed parton distributions: Method and two loop results.
Nucl. Phys. B, 574:347–406.

[Belitsky et al., 2000b] Belitsky, A. V., Freund, A., and Müller, D. (2000b).

NLO evolution kernels for skewed transversity distributions.
Phys. Lett. B, 493:341–349.

[Belitsky and Müller, 1998] Belitsky, A. V. and Müller, D. (1998).
Next-to-leading order evolution of twist-2 conformal operators: The Abelian case.
Nucl. Phys. B, 527:207–234.

References IX

[Belitsky and Müller, 1999a] Belitsky, A. V. and Müller, D. (1999a).
Broken conformal invariance and spectrum of anomalous dimensions in QCD.
Nucl. Phys. B, 537:397–442.

[Belitsky and Müller, 1999b] Belitsky, A. V. and Müller, D. (1999b).
Exclusive evolution kernels in two loop order: Parity even sector.
Phys. Lett. B, 464:249–256.

[Belitsky and Müller, 2000] Belitsky, A. V. and Müller, D. (2000).
Off forward gluonometry.
Phys. Lett. B, 486:369–377.

[Belitsky et al., 1999] Belitsky, A. V., Müller, D., and Freund, A. (1999).
Reconstruction of nonforward evolution kernels.
Phys. Lett. B, 461:270–279.

References X

[Bennett and Gracey, 1998] Bennett, J. F. and Gracey, J. A. (1998).
Determination of the anomalous dimension of gluonic operators in deep inelastic scattering at O (1/N(f)).
Nucl. Phys. B, 517:241–268.

[Bertone et al., 2023] Bertone, V., del Castillo, R. F., Echevarria, M. G., del Río, O., and Rodini, S. (2023).
One-Loop Evolution of Twist-2 Generalised Parton Distributions.

[Bertone et al., 2022] Bertone, V., Dutrieux, H., Mezrag, C., Morgado, J. M., and Moutarde, H. (2022).
Revisiting evolution equations for generalised parton distributions.
Eur. Phys. J. C, 82(10):888.

References XI

[Bierenbaum et al., 2009] Bierenbaum, I., Blumlein, J., and Klein, S. (2009).
Mellin Moments of the O(alpha**3(s)) Heavy Flavor Contributions to unpolarized Deep-Inelastic Scattering at Q**2 >> m**2 and Anomalous Dimensions.
Nucl. Phys. B, 820:417–482.

[Blümlein, 2001] Blümlein, J. (2001).
On the anomalous dimension of the transversity distribution $\mathrm{h}_1(\mathrm{x}, Q^2)$.
Eur. Phys. J. C, 20:683–687.

[Blumlein, 2004] Blumlein, J. (2004).
Mathematical structure of anomalous dimensions and QCD Wilson coefficients in higher order.
Nucl. Phys. B Proc. Suppl., 135:225–231.

References XII

[Blümlein, 2023] Blümlein, J. (2023).
Deep-Inelastic Scattering: What do we know ?

[Blumlein et al., 1997a] Blumlein, J., Geyer, B., and Robaschik, D. (1997a).
On the evolution kernels of twist-2 light ray operators for unpolarized and polarized deep inelastic scattering.
Phys. Lett. B, 406:161–170.

[Blumlein et al., 1997b] Blumlein, J., Geyer, B., and Robaschik, D. (1997b).
Twist-2 light ray operators: Anomalous dimensions and evolution equations.
In *2nd Topical Workshop on Deep Inelastic Scattering off Polarized Targets: Theory Meets Experiment (SPIN 97)*, pages 196–209.

References XIII

[Blümlein et al., 1999] Blümlein, J., Geyer, B., and Robaschik, D. (1999).
The Virtual Compton amplitude in the generalized Bjorken region: twist-2 contributions.
Nucl. Phys. B, 560:283–344.

[Blumlein et al., 2009] Blumlein, J., Kauers, M., Klein, S., and Schneider, C. (2009).
Determining the closed forms of the O(a**3(s)) anomalous dimensions and Wilson coefficients from Mellin moments by means of computer algebra.
Comput. Phys. Commun., 180:2143–2165.

[Blümlein and Kurth, 1999] Blümlein, J. and Kurth, S. (1999).
Harmonic sums and Mellin transforms up to two loop order.
Phys. Rev. D, 60:014018.

References XIV

[Blümlein et al., 2021] Blümlein, J., Marquard, P., Schneider, C., and Schönwald, K. (2021).
The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements.
Nucl. Phys. B, 971:115542.

[Blümlein et al., 2022a] Blümlein, J., Marquard, P., Schneider, C., and Schönwald, K. (2022a).
The 3-loop anomalous dimensions from off-shell operator matrix elements.
PoS, LL2022:048.

References XV

[Blümlein et al., 2022b] Blümlein, J., Marquard, P., Schneider, C., and Schönwald, K. (2022b).
The three-loop polarized singlet anomalous dimensions from off-shell operator matrix elements.
JHEP, 01:193.

[Blumlein et al., 1998a] Blumlein, J., Ravindran, V., and van Neerven, W. L. (1998a).
Relations among polarized and unpolarized splitting functions beyond leading order.
Acta Phys. Polon. B, 29:2581–2590.

References XVI

[Blumlein et al., 1998b] Blumlein, J., Ravindran, V., van Neerven, W. L., and Vogt, A. (1998b).
The Unpolarized gluon anomalous dimension at small x.
In *6th International Workshop on Deep Inelastic Scattering and QCD (DIS 98)*, pages 211–216.

[Blumlein and Vogt, 1998] Blumlein, J. and Vogt, A. (1998).
The resummed gluon anomalous dimension and structure functions at small x.
Phys. Rev. D, 57(1):R1.

[Boer et al., 2011] Boer, D. et al. (2011).
Gluons and the quark sea at high energies: Distributions, polarization, tomography.

References XVII

[Braun et al., 2022] Braun, V. M., Chetyrkin, K. G., and Manashov, A. N. (2022).
NNLO anomalous dimension matrix for twist-two flavor-singlet operators.
Phys. Lett. B, 834:137409.

[Braun et al., 2019a] Braun, V. M., Ji, Y., and Manashov, A. N. (2019a).
Two-loop evolution equation for the B-meson distribution amplitude.
Phys. Rev. D, 100(1):014023.

[Braun and Manashov, 2013] Braun, V. M. and Manashov, A. N. (2013).
Evolution equations beyond one loop from conformal symmetry.
Eur. Phys. J. C, 73:2544.

[Braun and Manashov, 2014] Braun, V. M. and Manashov, A. N. (2014).
Two-loop evolution equations for light-ray operators.
Phys. Lett. B, 734:137–143.

References XVIII

[Braun et al., 2016] Braun, V. M., Manashov, A. N., Moch, S., and Strohmaier, M. (2016).
Two-loop conformal generators for leading-twist operators in QCD.
JHEP, 03:142.

[Braun et al., 2017] Braun, V. M., Manashov, A. N., Moch, S., and Strohmaier, M. (2017).
Three-loop evolution equation for flavor-nonsinglet operators in off-forward kinematics.
JHEP, 06:037.

[Braun et al., 2019b] Braun, V. M., Manashov, A. N., Moch, S., and Strohmaier, M. (2019b).
Two-loop evolution equations for flavor-singlet light-ray operators.
JHEP, 02:191.

References XIX

[Braun et al., 2021] Braun, V. M., Manashov, A. N., Moch, S., and Strohmaier, M. (2021).
Three-loop off-forward evolution kernel for axial-vector operators in Larin's scheme.
Phys. Rev. D, 103(9):094018.

[Braun et al., 2019c] Braun, V. M., Manashov, A. N., Moch, S. O., and Strohmaier, M. (2019c).
Conformal symmetry of QCD in *d*-dimensions.
Phys. Lett. B, 793:78–84.

[Braunschweig et al., 1986] Braunschweig, T., Geyer, B., Horejsi, J., and Robaschik, D. (1986).
Hadron Operators on the Light Cone.
Z. Phys. C, 33:275.

References XX

[Chekanov et al., 2003] Chekanov, S. et al. (2003).
Measurement of deeply virtual Compton scattering at HERA.
Phys. Lett. B, 573:46–62.

[Cortes et al., 1992] Cortes, J. L., Pire, B., and Ralston, J. P. (1992).
Measuring the transverse polarization of quarks in the proton.
Z. Phys. C, 55:409–416.

[Curci et al., 1980] Curci, G., Furmanski, W., and Petronzio, R. (1980).
Evolution of Parton Densities Beyond Leading Order: The Nonsinglet Case.
Nucl. Phys. B, 175:27–92.

[Davies et al., 2017] Davies, J., Vogt, A., Ruijl, B., Ueda, T., and Vermaseren, J. A. M. (2017).
Large-n_f contributions to the four-loop splitting functions in QCD.
Nucl. Phys. B, 915:335–362.

References XXI

[Deur et al., 2018] Deur, A., Brodsky, S. J., and De Téramond, G. F. (2018).
The Spin Structure of the Nucleon.

[Diehl, 2003] Diehl, M. (2003).
Generalized parton distributions.
Phys. Rept., 388:41–277.

[Dittes et al., 1988] Dittes, F. M., Müller, D., Robaschik, D., Geyer, B., and Hořejši, J. (1988).
The Altarelli-Parisi Kernel as Asymptotic Limit of an Extended Brodsky-Lepage Kernel.
Phys. Lett. B, 209:325–329.

References XXII

[Dokshitzer, 1977] Dokshitzer, Y. L. (1977).
Calculation of the Structure Functions for Deep Inelastic Scattering and e^+e^- Annihilation by Perturbation Theory in Quantum Chromodynamics.
Sov. Phys. JETP, 46:641–653.

[Efremov and Radyushkin, 1980] Efremov, A. V. and Radyushkin, A. V. (1980).
Asymptotical Behavior of Pion Electromagnetic Form-Factor in QCD.
Theor. Math. Phys., 42:97–110.

[Falcioni and Herzog, 2022] Falcioni, G. and Herzog, F. (2022).
Renormalization of gluonic leading-twist operators in covariant gauges.
JHEP, 05:177.

References XXIII

[Falcioni et al., 2023a] Falcioni, G., Herzog, F., Moch, S., Vermaseren, J., and Vogt, A. (2023a).
The double fermionic contribution to the four-loop quark-to-gluon splitting function.
Phys. Lett. B, 848:138351.

[Falcioni et al., 2023b] Falcioni, G., Herzog, F., Moch, S., and Vogt, A. (2023b).
Four-loop splitting functions in QCD – The gluon-to-quark case.
Phys. Lett. B, 846:138215.

[Falcioni et al., 2023c] Falcioni, G., Herzog, F., Moch, S., and Vogt, A. (2023c).
Four-loop splitting functions in QCD – The quark-quark case.
Phys. Lett. B, 842:137944.

References XXIV

[Floratos et al., 1977] Floratos, E. G., Ross, D. A., and Sachrajda, C. T. (1977).
Higher Order Effects in Asymptotically Free Gauge Theories: The Anomalous Dimensions of Wilson Operators.
Nucl. Phys. B, 129:66–88.
[Erratum: Nucl.Phys.B 139, 545–546 (1978)].

[Floratos et al., 1979] Floratos, E. G., Ross, D. A., and Sachrajda, C. T. (1979).
Higher Order Effects in Asymptotically Free Gauge Theories. 2. Flavor Singlet Wilson Operators and Coefficient Functions.
Nucl. Phys. B, 152:493–520.

[Furmanski and Petronzio, 1980] Furmanski, W. and Petronzio, R. (1980).
Singlet Parton Densities Beyond Leading Order.
Phys. Lett. B, 97:437–442.

References XXV

[Gehrmann et al., 2023a] Gehrmann, T., von Manteuffel, A., Sotnikov, V., and Yang, T.-Z. (2023a).
Complete N_f^2 contributions to four-loop pure-singlet splitting functions.

[Gehrmann et al., 2023b] Gehrmann, T., von Manteuffel, A., Sotnikov, V., and Yang, T.-Z. (2023b).
The $N_f\, C_F^3$ contribution to the non-singlet splitting function at four-loop order.

[Gehrmann et al., 2023c] Gehrmann, T., von Manteuffel, A., and Yang, T.-Z. (2023c).
Renormalization of twist-two operators in covariant gauge to three loops in QCD.
JHEP, 04:041.

References XXVI

[Geyer, 1982] Geyer, B. (1982).
Anomalous dimensions in local and non-local light cone expansion. (Talk).
Czech. J. Phys. B, 32:645–652.

[Geyer et al., 1985] Geyer, B., Robaschik, D., Bordag, M., and Horejsi, J. (1985).
NONLOCAL LIGHT CONE EXPANSIONS AND EVOLUTION EQUATIONS.
Z. Phys. C, 26:591–600.

[Göckeler et al., 2005] Göckeler, M., Horsley, R., Pleiter, D., Rakow, P. E. L., and Schierholz, G. (2005).
A Lattice determination of moments of unpolarised nucleon structure functions using improved Wilson fermions.
Phys. Rev. D, 71:114511.

References XXVII

[Gonzalez-Arroyo and Lopez, 1980] Gonzalez-Arroyo, A. and Lopez, C. (1980).
Second Order Contributions to the Structure Functions in Deep Inelastic Scattering. 3. The Singlet Case.
Nucl. Phys. B, 166:429–459.

[Gonzalez-Arroyo et al., 1979] Gonzalez-Arroyo, A., Lopez, C., and Yndurain, F. J. (1979).
Second Order Contributions to the Structure Functions in Deep Inelastic Scattering. 1. Theoretical Calculations.
Nucl. Phys. B, 153:161–186.

References XXVIII

[Gonzalez-Arroyo et al., 1980] Gonzalez-Arroyo, A., Lopez, C., and Yndurain, F. J. (1980).
Testing QCD Predictions Beyond the Leading Order for Nonsinglet Structure Functions in Neutrino Scattering.
Nucl. Phys. B, 174:474–490.

[Gracey, 1994] Gracey, J. A. (1994).
Anomalous dimension of nonsinglet Wilson operators at $O(1/n_f)$ in deep inelastic scattering.
Phys. Lett. B, 322:141–146.

[Gracey, 1996] Gracey, J. A. (1996).
Anomalous dimensions of operators in polarized deep inelastic scattering at O(1/N(f)).
Nucl. Phys. B, 480:73–98.

References XXIX

[Gracey, 2003a] Gracey, J. A. (2003a).
Three loop anomalous dimension of nonsinglet quark currents in the RI-prime scheme.
Nucl. Phys. B, 662:247–278.

[Gracey, 2003b] Gracey, J. A. (2003b).
Three loop anomalous dimension of the second moment of the transversity operator in the MS-bar and RI-prime schemes.
Nucl. Phys. B, 667:242–260.

[Gracey, 2006a] Gracey, J. A. (2006a).
Three loop anomalous dimensions of higher moments of the non-singlet twist-2 Wilson and transversity operators in the anti-MS and RI-prime schemes.
JHEP, 10:040.

References XXX

[Gracey, 2006b] Gracey, J. A. (2006b).
Three loop MS-bar transversity operator anomalous dimensions for fixed moment n <= 8.
Phys. Lett. B, 643:374–378.

[Gracey, 2009] Gracey, J. A. (2009).
Three loop anti-MS operator correlation functions for deep inelastic scattering in the chiral limit.
JHEP, 04:127.

[Gracey, 2011a] Gracey, J. A. (2011a).
Amplitudes for the n = 3 moment of the Wilson operator at two loops in the RI/'SMOM scheme.
Phys. Rev. D, 84:016002.

References XXXI

[Gracey, 2011b] Gracey, J. A. (2011b).
Two loop renormalization of the $n = 2$ Wilson operator in the RI'/SMOM scheme.
JHEP, 03:109.

[Graham et al., 1989] Graham, R. L., Knuth, D. E., and Patashnik, O. (1989).
Concrete mathematics - a foundation for computer science.
Addison-Wesley.

[Gribov et al., 1983] Gribov, L. V., Levin, E. M., and Ryskin, M. G. (1983).
Semihard Processes in QCD.
Phys. Rept., 100:1–150.

References XXXII

[Gribov and Lipatov, 1972] Gribov, V. N. and Lipatov, L. N. (1972).
Deep inelastic e p scattering in perturbation theory.
Sov. J. Nucl. Phys., 15:438–450.

[Gross and Wilczek, 1973] Gross, D. J. and Wilczek, F. (1973).
Asymptotically Free Gauge Theories - I.
Phys. Rev. D, 8:3633–3652.

[Gross and Wilczek, 1974] Gross, D. J. and Wilczek, F. (1974).
ASYMPTOTICALLY FREE GAUGE THEORIES. 2.
Phys. Rev. D, 9:980–993.

[Hayashigaki et al., 1997] Hayashigaki, A., Kanazawa, Y., and Koike, Y. (1997).
Next-to-leading order q^2 evolution of the transversity distribution $h_1(x,q^2)$.
Phys. Rev. D, 56:7350–7360.

References XXXIII

[Herzog et al., 2019] Herzog, F., Moch, S., Ruijl, B., Ueda, T., Vermaseren, J. A. M., and Vogt, A. (2019).
Five-loop contributions to low-N non-singlet anomalous dimensions in QCD.
Phys. Lett. B, 790:436–443.

[Herzog et al., 2016] Herzog, F., Ruijl, B., Ueda, T., Vermaseren, J. A. M., and Vogt, A. (2016).
FORM, Diagrams and Topologies.
PoS, LL2016:073.

[Hoodbhoy and Ji, 1998] Hoodbhoy, P. and Ji, X.-D. (1998).
Helicity flip off forward parton distributions of the nucleon.
Phys. Rev. D, 58:054006.

References XXXIV

[Jaffe and Ji, 1991] Jaffe, R. L. and Ji, X.-D. (1991).
Chiral odd parton distributions and polarized Drell-Yan.
Phys. Rev. Lett., 67:552–555.

[Jaffe and Ji, 1992] Jaffe, R. L. and Ji, X.-D. (1992).
Chiral odd parton distributions and Drell-Yan processes.
Nucl. Phys. B, 375:527–560.

[Ji et al., 2021] Ji, X., Yuan, F., and Zhao, Y. (2021).
What we know and what we don't know about the proton spin after 30 years.
Nature Rev. Phys., 3(1):27–38.

[Ji, 1997] Ji, X.-D. (1997).
Deeply virtual Compton scattering.
Phys. Rev. D, 55:7114–7125.

[Ji et al., 2023] Ji, Y., Manashov, A., and Moch, S.-O. (2023).
Evolution kernels of twist-two operators.
Phys. Rev. D, 108(5):054009.

[Kaiser, 2012] Kaiser, R. (2012).
The 3D structure of the proton.
J. Phys. Conf. Ser., 381:012018.

[Kauers and Paule, 2011] Kauers, M. and Paule, P. (2011).
The Concrete Tetrahedron - Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates.
Texts & Monographs in Symbolic Computation. Springer.

[Kisselev, 2012] Kisselev, A. V. (2012).
Operator product expansion coefficient functions in terms of composite operators only: Singlet case.
Phys. Rev. D, 85:094022.

References XXXVI

[Kisselev and Petrov, 2005] Kisselev, A. V. and Petrov, V. A. (2005).
OPE coefficient functions in terms of composite operators only: Nonsinglet case.
Phys. Rev. D, 71:085020.

[Klein, 2009] Klein, S. W. G. (2009).
Mellin Moments of Heavy Flavor Contributions to $F_2(x, Q^2)$ at NNLO.
PhD thesis, Dortmund U., Berlin.

[Kniehl and Velizhanin, 2023] Kniehl, B. A. and Velizhanin, V. N. (2023).
Anomalous dimensions of twist-two operators in extended N=2 and N=4 super Yang-Mills theories.

[Kniehl and Veretin, 2020] Kniehl, B. A. and Veretin, O. L. (2020).
Moments $n = 2$ and $n = 3$ of the Wilson twist-two operators at three loops in the RI$'$/SMOM scheme.
Nucl. Phys. B, 961:115229.

[Kuipers et al., 2013] Kuipers, J., Ueda, T., Vermaseren, J. A. M., and Vollinga, J. (2013).
FORM version 4.0.
Comput. Phys. Commun., 184:1453–1467.

[Kumano and Miyama, 1997] Kumano, S. and Miyama, M. (1997).
Two loop anomalous dimensions for the structure function h1.
Phys. Rev. D, 56:R2504–R2508.

[Larin et al., 1997] Larin, S. A., Nogueira, P., van Ritbergen, T., and Vermaseren, J. A. M. (1997).
The Three loop QCD calculation of the moments of deep inelastic structure functions.
Nucl. Phys. B, 492:338–378.

References XXXVIII

[Leader and Lorcé, 2014] Leader, E. and Lorcé, C. (2014).
The angular momentum controversy: What's it all about and does it matter?
Phys. Rept., 541(3):163–248.

[Lepage and Brodsky, 1980] Lepage, G. P. and Brodsky, S. J. (1980).
Exclusive Processes in Perturbative Quantum Chromodynamics.
Phys. Rev. D, 22:2157.

[Makeenko, 1981] Makeenko, Y. M. (1981).
Conformal Operators in Quantum Chromodynamics.
Sov. J. Nucl. Phys., 33:440.

References XXXIX

[Manashov and Strohmaier, 2015] Manashov, A. N. and Strohmaier, M. (2015).
Conformal constraints for anomalous dimensions of leading twist operators.
Eur. Phys. J. C, 75(8):363.

[Martin and Ryskin, 1998] Martin, A. D. and Ryskin, M. G. (1998).
The effect of off diagonal parton distributions in diffractive vector meson electroproduction.
Phys. Rev. D, 57:6692–6700.

[Mertig and van Neerven, 1996] Mertig, R. and van Neerven, W. L. (1996).
The Calculation of the two loop spin splitting functions $P_{ij}^{(1)}(x)$.
Z. Phys. C, 70:637–654.

References XL

[Mikhailov and Vladimirov, 2009a] Mikhailov, S. and Vladimirov, A. (2009a).
Erbl and dglap kernels for transversity distributions. two-loop calculations in covariant gauge.
Physics Letters B, 671(1):111–118.

[Mikhailov, 1998a] Mikhailov, S. V. (1998a).
Renormalon chains contributions to nonsinglet evolutional kernels in QCD.
Phys. Lett. B, 431:387–394.

[Mikhailov, 1998b] Mikhailov, S. V. (1998b).
Renormalon chains contributions to the nonsinglet evolution kernels in ϕ^3 in six-dimensions and QCD.
Phys. Lett. B, 416:421–432.

References XLI

[Mikhailov, 2000] Mikhailov, S. V. (2000).
A multiloop improvement of nonsinglet QCD evolution equations.
Phys. Rev. D, 62:034002.

[Mikhailov and Vladimirov, 2009b] Mikhailov, S. V. and Vladimirov, A. A. (2009b).
ERBL and DGLAP kernels for transversity distributions. Two-loop calculations in covariant gauge.
Phys. Lett. B, 671:111–118.

[Mikhailov and Volchanskiy, 2020] Mikhailov, S. V. and Volchanskiy, N. (2020).
Correlators of vector, tensor, and scalar composite vertices of order $O(\alpha_s^2 \beta_0)$.
JHEP, 21:197.

References XLII

[Moch et al., 2023] Moch, S., Ruijl, B., Ueda, T., Vermaseren, J., and Vogt, A. (2023).
Additional moments and x-space approximations of four-loop splitting functions in QCD.

[Moch et al., 2017] Moch, S., Ruijl, B., Ueda, T., Vermaseren, J. A. M., and Vogt, A. (2017).
Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond.
JHEP, 10:041.

[Moch et al., 2018] Moch, S., Ruijl, B., Ueda, T., Vermaseren, J. A. M., and Vogt, A. (2018).
On quartic colour factors in splitting functions and the gluon cusp anomalous dimension.
Phys. Lett. B, 782:627–632.

References XLIII

[Moch et al., 2022] Moch, S., Ruijl, B., Ueda, T., Vermaseren, J. A. M., and Vogt, A. (2022).
Low moments of the four-loop splitting functions in QCD.
Phys. Lett. B, 825:136853.

[Moch and Van Thurenhout, 2021] Moch, S. and Van Thurenhout, S. (2021).
Renormalization of non-singlet quark operator matrix elements for off-forward hard scattering.
Nucl. Phys. B, 971:115536.

[Moch et al., 2004] Moch, S., Vermaseren, J. A. M., and Vogt, A. (2004).
The Three loop splitting functions in QCD: The Nonsinglet case.
Nucl. Phys. B, 688:101–134.

References XLIV

[Moch et al., 2014] Moch, S., Vermaseren, J. A. M., and Vogt, A. (2014).
The Three-Loop Splitting Functions in QCD: The Helicity-Dependent Case.
Nucl. Phys. B, 889:351–400.

[Müller, 1991] Müller, D. (1991).
Constraints for anomalous dimensions of local light cone operators in ϕ^3 in six-dimensions theory.
Z. Phys. C, 49:293–300.

[Müller, 1994] Müller, D. (1994).
Conformal constraints and the evolution of the nonsinglet meson distribution amplitude.
Phys. Rev. D, 49:2525–2535.

References XLV

[Müller et al., 1994a] Müller, D., Robaschik, D., Geyer, B., Dittes, F. M., and Hořejši, J. (1994a).
Wave functions, evolution equations and evolution kernels from light ray operators of QCD.
Fortsch. Phys., 42:101–141.

[Müller et al., 1994b] Müller, D., Robaschik, D., Geyer, B., Dittes, F. M., and Hořejši, J. (1994b).
Wave functions, evolution equations and evolution kernels from light ray operators of QCD.
Fortsch. Phys., 42:101–141.

[Nogueira, 1993] Nogueira, P. (1993).
Automatic Feynman graph generation.
J. Comput. Phys., 105:279–289.

References XLVI

[Pasquini and Boffi, 2008] Pasquini, B. and Boffi, S. (2008).
The Structure of the nucleon from generalized parton distributions.
PoS, LC2008:014.

[Radyushkin, 1996] Radyushkin, A. V. (1996).
Scaling limit of deeply virtual Compton scattering.
Phys. Lett. B, 380:417–425.

[Radyushkin, 1997] Radyushkin, A. V. (1997).
Nonforward parton distributions.
Phys. Rev. D, 56:5524–5557.

[Radyushkin, 1999] Radyushkin, A. V. (1999).
Double distributions and evolution equations.
Phys. Rev. D, 59:014030.

References XLVII

[Ralston and Soper, 1979] Ralston, J. P. and Soper, D. E. (1979).
Production of Dimuons from High-Energy Polarized Proton Proton Collisions.
Nucl. Phys. B, 152:109.

[Ruijl et al., 2020] Ruijl, B., Ueda, T., and Vermaseren, J. A. M. (2020).
Forcer, a FORM program for the parametric reduction of four-loop massless propagator diagrams.
Comput. Phys. Commun., 253:107198.

[Ruijl et al., 2016] Ruijl, B., Ueda, T., Vermaseren, J. A. M., Davies, J., and Vogt, A. (2016).
First Forcer results on deep-inelastic scattering and related quantities.
PoS, LL2016:071.

References XLVIII

[Schneider, 2007] Schneider, C. (2007).
Symbolic summation assists combinatorics.
Seminaire Lotharingien de Combinatoire, 56:1–36.

[Schneider, 2013] Schneider, C. (2013).
Simplifying Multiple Sums in Difference Fields.
In *LHCPhenoNet School: Integration, Summation and Special Functions in Quantum Field Theory*, pages 325–360.

[Shifman and Vysotsky, 1981] Shifman, M. A. and Vysotsky, M. I. (1981).
Form-Factors of Heavy Mesons in QCD.
Nucl. Phys. B, 186:475–518.

[Soar et al., 2010] Soar, G., Moch, S., Vermaseren, J. A. M., and Vogt, A. (2010).
On Higgs-exchange DIS, physical evolution kernels and fourth-order splitting functions at large x.
Nucl. Phys. B, 832:152–227.

[Somogyi and Van Thurenhout, 2024] Somogyi, G. and Van Thurenhout, S. (2024).
All-order Feynman rule for leading-twist quark operators.

[Stepanyan et al., 2001] Stepanyan, S. et al. (2001).
Observation of exclusive deeply virtual Compton scattering in polarized electron beam asymmetry measurements.
Phys. Rev. Lett., 87:182002.

References L

[van Neerven and Vogt, 2000] van Neerven, W. L. and Vogt, A. (2000).
Improved approximations for the three loop splitting functions in QCD.
Phys. Lett. B, 490:111–118.

[van Ritbergen et al., 1999] van Ritbergen, T., Schellekens, A. N., and Vermaseren, J. A. M. (1999).
Group theory factors for Feynman diagrams.
Int. J. Mod. Phys. A, 14:41–96.

[Van Thurenhout, 2022] Van Thurenhout, S. (2022).
Off-forward anomalous dimensions of non-singlet transversity operators.
Nucl. Phys. B, 980:115835.

[Van Thurenhout, 2024] Van Thurenhout, S. (2024).
Basis transformation properties of anomalous dimensions for hard exclusive processes.
Nucl. Phys. B, 1000:116464.

References LI

[Van Thurenhout and Moch, 2022] Van Thurenhout, S. and Moch, S.-O. (2022).
Off-forward anomalous dimensions in the leading-n_f limit.
PoS, LL2022:076.

[Velizhanin, 2012a] Velizhanin, V. N. (2012a).
Four loop anomalous dimension of the second moment of the non-singlet twist-2 operator in QCD.
Nucl. Phys. B, 860:288–294.

[Velizhanin, 2012b] Velizhanin, V. N. (2012b).
Three loop anomalous dimension of the non-singlet transversity operator in QCD.
Nucl. Phys. B, 864:113–140.

References LII

[Velizhanin, 2020] Velizhanin, V. N. (2020).
Four-loop anomalous dimension of the third and fourth moments of the nonsinglet twist-2 operator in QCD.
Int. J. Mod. Phys. A, 35(32):2050199.

[Vermaseren, 1999] Vermaseren, J. A. M. (1999).
Harmonic sums, Mellin transforms and integrals.
Int. J. Mod. Phys. A, 14:2037–2076.

[Vermaseren, 2000] Vermaseren, J. A. M. (2000).
New features of FORM.

[Vogelsang, 1998] Vogelsang, W. (1998).
Next-to-leading order evolution of transversity distributions and Soffer's inequality.
Phys. Rev. D, 57:1886–1894.

References LIII

[Vogt et al., 2018] Vogt, A., Herzog, F., Moch, S., Ruijl, B., Ueda, T., and Vermaseren, J. A. M. (2018).
Anomalous dimensions and splitting functions beyond the next-to-next-to-leading order.
PoS, LL2018:050.

[Vogt et al., 2010a] Vogt, A., Moch, S., Soar, G., and Vermaseren, J. A. M. (2010a).
Higher-Order Predictions from Physical Evolution Kernels.
PoS, RADCOR2009:053.

[Vogt et al., 2004] Vogt, A., Moch, S., and Vermaseren, J. A. M. (2004).
The Three-loop splitting functions in QCD: The Singlet case.
Nucl. Phys. B, 691:129–181.

References LIV

[Vogt et al., 2010b] Vogt, A., Soar, G., Moch, S., and Vermaseren, J. A. M. (2010b).
On higher-order flavour-singlet splitting and coefficient functions at large x.
PoS, DIS2010:139.

[Zeilberger, 1991] Zeilberger, D. (1991).
The method of creative telescoping.
Journal of Symbolic Computation, 11(3):195–204.