Form factors from the staircase

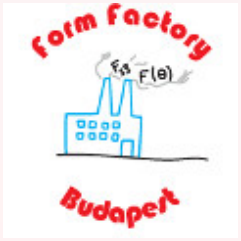
Gábor Takács

Department of Theoretical Physics
Budapest University of Technology and Economics

MTA-BME "Momentum" Statistical Field Theory Research Group

P. Dorey, G. Siviour and G. Takács, JHEP, in press. arXiv:1412.4787 [hep-th]

ELTE Theoretical Physics Seminar
Budapest, 6th March, 2015



Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Outline

1. Introduction: Zamolodchikov's roaming flows
   - Roaming flows from sinh-Gordon TBA
   - Flows between minimal models
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Outline

1. Introduction: Zamolodchikov's roaming flows
   - Roaming flows from sinh-Gordon TBA
   - Flows between minimal models
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Zamolodchikov's roaming flows

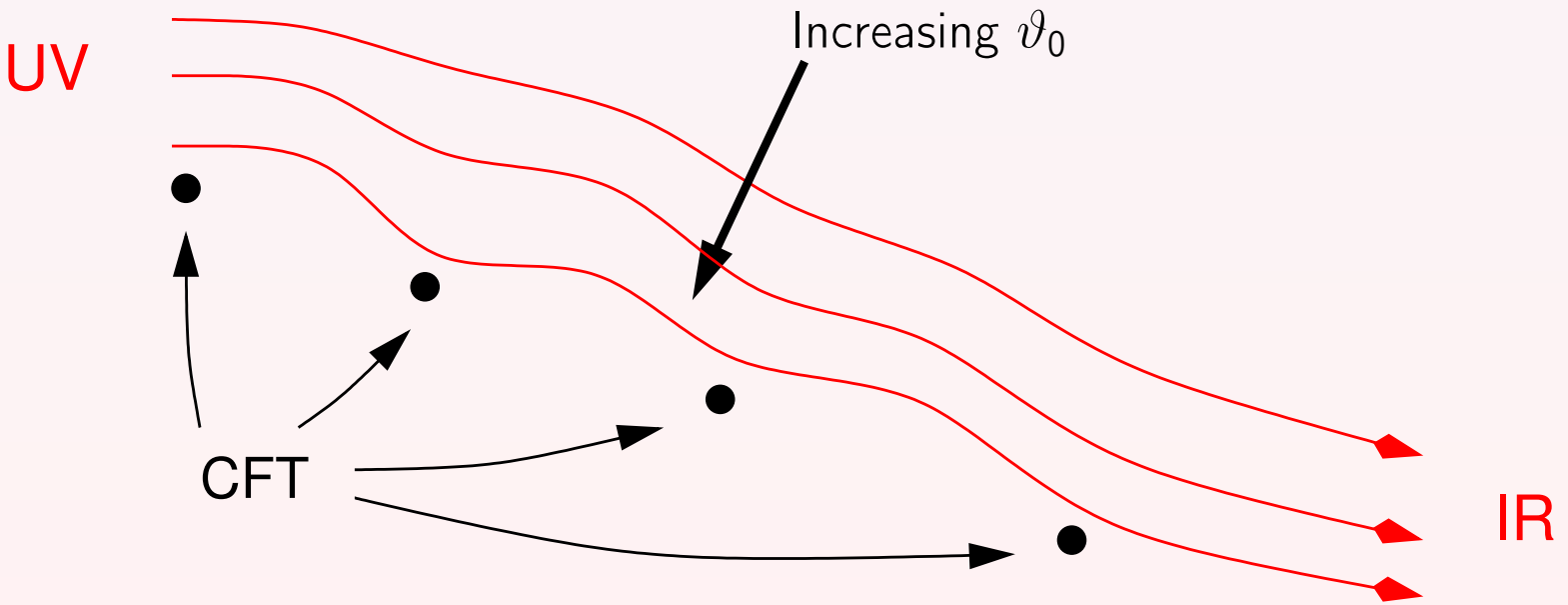
The sinh-Gordon model

$$\mathcal{A} = \int d^2x \left(\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{M^2}{b^2} \cosh b\Phi \right)$$

$$\gamma = \frac{\pi b^2}{8\pi + b^2} \quad \text{Duality: } \gamma \to \pi - \gamma$$

Roaming continuation

$$\gamma = \frac{\pi}{2} \pm i\vartheta_0$$



The sinh-Gordon TBA system

Sinh-Gordon TBA

$$\epsilon(\theta) = mR\cosh\theta - \int_{-\infty}^{\infty} d\theta' \phi(\theta-\theta') \log\left(1+e^{-\epsilon(\theta)}\right)$$

$$\phi(\theta) = -\frac{i}{2\pi}\frac{d}{d\theta}\log S(\theta) \qquad S(\theta) = \frac{\sinh\theta - i\sin\gamma}{\sinh\theta + i\sin\gamma}$$

Free energy

$$F(R) = -\frac{\pi c_{eff}(r)}{6R} r = mR$$

Effective central charge:

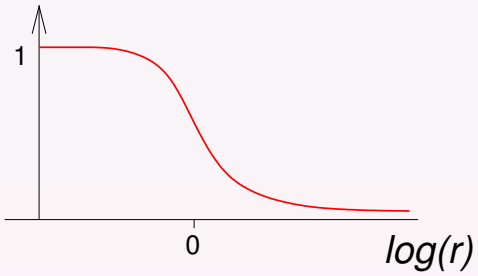
$$c_{eff}(r) = \frac{3}{\pi^2}\int_{-\infty}^{\infty} d\theta r\cosh(\theta)\log\left(1+e^{-\epsilon(\theta)}\right)$$

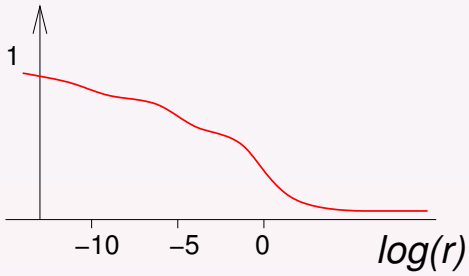
The roaming flow in TBA

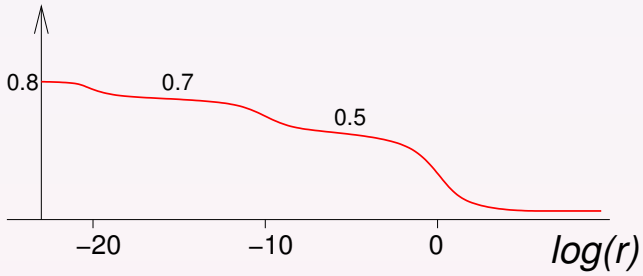
Roaming continuation (Alyosha Zamolodchikov, 1991)

$$\gamma = \frac{\pi}{2} \pm i\vartheta_0$$

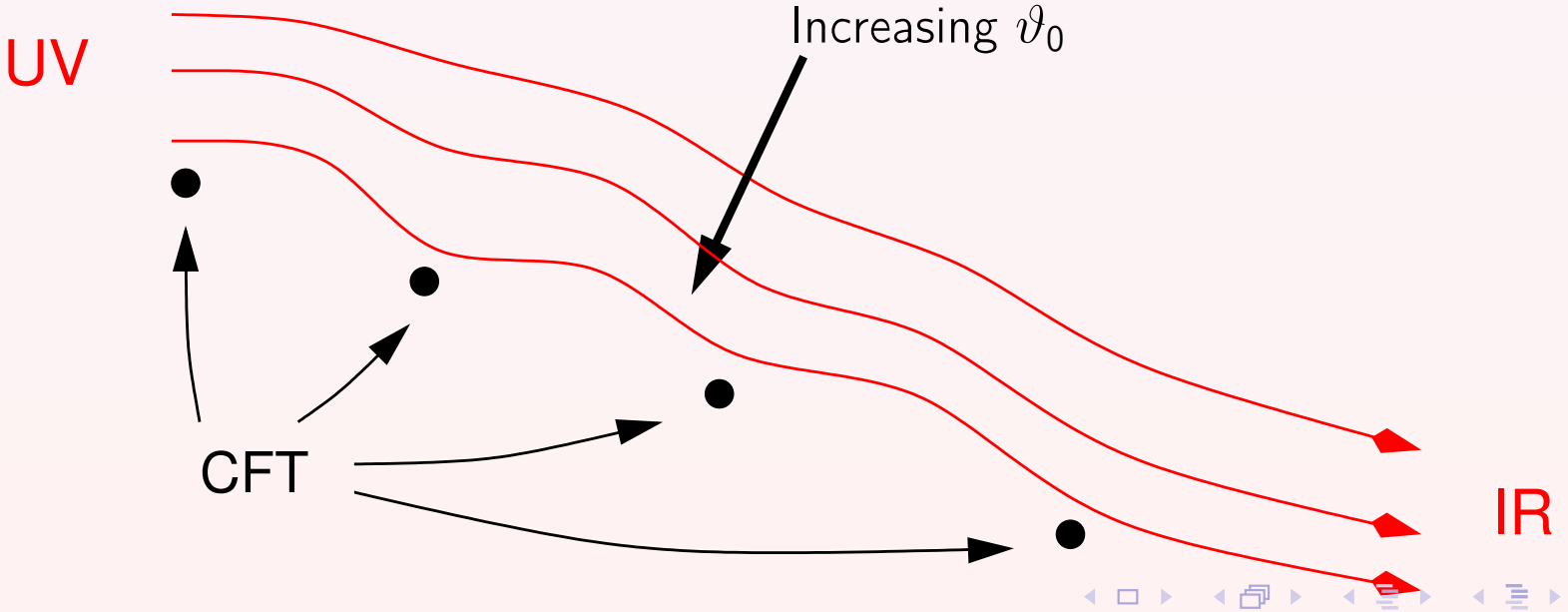
Plots for c_{eff}:







$\vartheta_0 = 0$ $\vartheta_0 = 10$ $\vartheta_0 = 20$



Outline

1. Introduction: Zamolodchikov's roaming flows
   - Roaming flows from sinh-Gordon TBA
   - Flows between minimal models
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Flows between minimal models

Minimal models of conformal field theory

$$\mathcal{M}_p \quad : \quad c = 1 - \frac{6}{p(p+1)}$$

$$\text{primaries} \quad : \quad h_{r,s} = \frac{((p+1)r - ps)^2 - 1}{4p(p+1)}$$

Massless flow

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_* + \lambda \int dtdx \Phi_{1,3}(x) \\
& \qquad g = \lambda a^{2-2h_{1,3}} \qquad h_{1,3} = 1 - \frac{2}{p+1} \\
\beta(g) &= (2 - 2h_{1,3})g - Cg^2 + O(g^3) \\
\Rightarrow g^* &= \frac{(2 - 2h_{1,3})}{C} \propto \frac{1}{p+1} \ll 1 \quad \text{for large } p
\end{aligned}$$

Flows between minimal models

Large p: fixed point is in perturbative regime
$\Rightarrow$ ϵ-expansion with $\epsilon = 1 - h_{1,3} = \frac{1}{p+1}$

$$\begin{array}{lcll} \mathcal{M}_p & \stackrel{\Phi_{1,3}}{\longrightarrow} & \mathcal{M}_{p-1} & \\ \Phi_{1,3} & \longrightarrow & \Phi_{3,1} & \text{(valid except } p = 4 : \ T\bar{T}) \\ \Phi_{3,1} & \longrightarrow & \Phi_{1,3} & \end{array}$$

IR action

$$\mathcal{A}_{p-1} + \lambda \int dtdx \Phi_{3,1}(x) \qquad h_{3,1} = 1 + \frac{2}{p}$$

Last flow: $p = 3$ (Ising) leads to free massive Majorana fermion

$$\mathcal{A}_* + \lambda \int dtdx \Phi_{1,3}(x) = \int d^2x \bar{\Psi} \left(i\not{\partial} - m \right) \Psi \quad \Rightarrow \quad c_{IR} = 0$$

Flows between minimal models

Intuitively: roaming flow is

$$\cdots \to \mathcal{A}_p + \lambda_p \int \Phi_{13} + \mu_p \int \Phi_{31} \to \mathcal{A}_{p-1} + \lambda_{p-1} \int \Phi_{31} + \mu_{p-1} \int \Phi_{13} \to \cdots$$

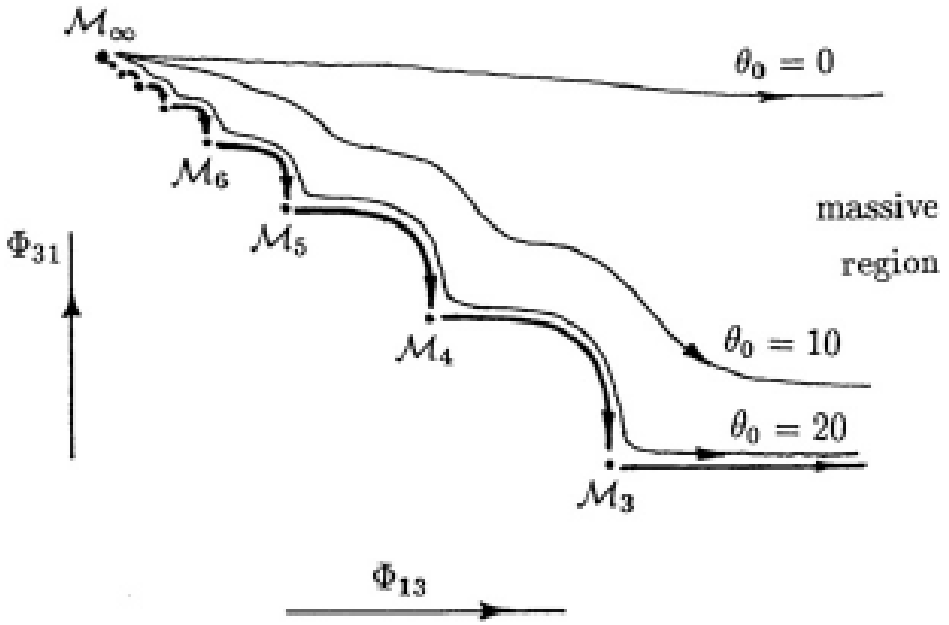
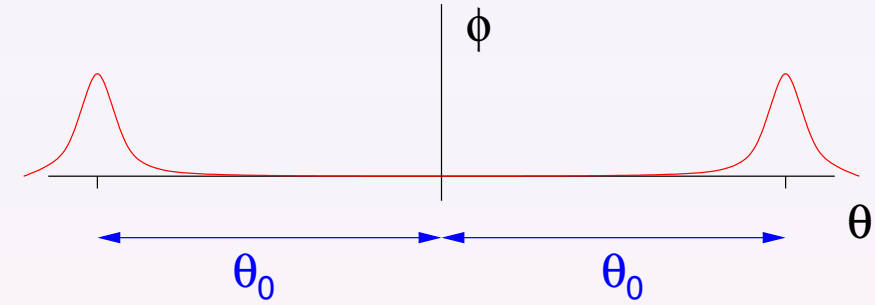


Figure 6. Sequence of $\mathcal{M}_p$ fixed points connected by integrable interpolating trajectories $\mathcal{M}_p \to \mathcal{M}_{p-1}$ is drawn schematically. At $\theta_0 \to \infty$ the wandering trajectory flows arbitrary close near the sequence.

From: Al. B. Zamolodchikov, J. Phys. A: Math. Gen. 39 (2006) 12847

TBA for flows between minimal models

$$\phi(\theta) = \frac{1}{2\pi}\left(\frac{1}{\cosh(\theta-\vartheta_0)} + \frac{1}{\cosh(\theta+\vartheta_0)}\right)$$



Scaling to step k ($\mathcal{M}_{k+2} \to \mathcal{M}_{k+1}$)

$$\begin{aligned} r &= \tilde{r} e^{-(k-1)\vartheta_0/2} \\ \epsilon_j(\theta) &= \epsilon\left(\theta + \frac{k-2j+1}{2}\vartheta_0\right) \qquad j = 1,\ldots,k \end{aligned}$$

leads to the (RSOS) TBA

$$\epsilon_1(\theta) = \frac{\tilde{r}}{2}e^{\theta} - \int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\log\left(1+e^{-\epsilon_2(\theta')}\right)$$

$$\epsilon_j(\theta) = -\int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\left\{\log\left(1+e^{-\epsilon_{j-1}(\theta')}\right) + \log\left(1+e^{-\epsilon_{j+1}(\theta')}\right)\right\}$$

$$j = 2,\ldots,k-1$$

$$\epsilon_k(\theta) = \frac{\tilde{r}}{2}e^{-\theta} - \int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\log\left(1+e^{-\epsilon_{k-1}(\theta')}\right)$$

TBA for flows between minimal models

RSOS TBA (Alyosha Zamolodchikov, 1990-91)

$$
\begin{aligned}
\epsilon_1(\theta) &= \frac{\tilde{r}}{2}e^{\theta} - \int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\log\left(1+e^{-\epsilon_2(\theta')}\right) \\
\epsilon_j(\theta) &= -\int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\left\{\log\left(1+e^{-\epsilon_{j-1}(\theta')}\right)+\log\left(1+e^{-\epsilon_{j+1}(\theta')}\right)\right\} \\
& \qquad j=2,\ldots,k-1 \\
\epsilon_k(\theta) &= \frac{\tilde{r}}{2}e^{-\theta} - \int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\log\left(1+e^{-\epsilon_{k-1}(\theta')}\right)
\end{aligned}
$$

$$
c_{eff}(\tilde{r}) = \frac{3}{\pi^2}\int_{-\infty}^{\infty} d\theta \frac{\tilde{r}}{2}\left\{e^{\theta}\log\left(1+e^{-\epsilon_1(\theta)}\right)+e^{-\theta}\log\left(1+e^{-\epsilon_k(\theta)}\right)\right\}
$$

describes the flow

$$
\mathcal{M}_{k+2} \xrightarrow{\Phi_{1,3}} \mathcal{M}_{k+1} \qquad k \geq 2
$$

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
   - Zamolodchikov's c-theorem
   - The Ising and tricritical Ising steps
   - The first two steps
   - A first look into relocalisation
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
   - Zamolodchikov's c-theorem
   - The Ising and tricritical Ising steps
   - The first two steps
   - A first look into relocalisation
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

C-theorem for two-dimensional QFT

C-theorem (Sasha Zamolodchikov, 1987): there is a $c(R)$ such that

$$\frac{dc(R)}{dR} = -\frac{3}{2}R^3\langle\Theta(R)\Theta(0)\rangle < 0$$

where Θ is the trace of the stress-energy tensor;
and in conformal regime(s) $c(R)$ gives the CFT central charge.

Integrated form (John Cardy, 1988)

$$c(R) = c(\infty) + \int_R^\infty dR' \frac{3}{2}\left(R'\right)^3 \langle\Theta(R')\Theta(0)\rangle$$

In general, apart from conformal regime(s)

$$\begin{array}{ccc} c(R) & \neq & c_{eff}(R) \\ \text{correlators} & & \text{TBA} \\ R: \text{ distance} & & R: \text{ volume} \end{array}$$

However:

- both reduce to the central charge in conformal regime(s)
- both decrease with R (for unitary theories)

C-theorem sum rule

Using form factors

$$\langle \Theta(R)\Theta(0)\rangle = \sum_{n=0}^{\infty} \int \frac{d\theta_1 \dots d\theta_n}{(2\pi)^n n!} |F_n^{\Theta}(\theta_1 \dots \theta_n)|^2 \, e^{-mR\sum_{i=1}^{n}\cosh\theta_i}$$

$$F_n^{\Theta} = \langle 0|\Theta(0)|\theta_1, \dots \theta_n\rangle$$

leads to the c-theorem sum rule

$$c(r) = 3\sum_{n=0}^{\infty} \int \frac{d\theta_1 \dots d\theta_n}{(2\pi)^n n!} \times \left\{ \frac{6 + 6rE + 3r^2E^2 + r^3E^3}{2E^4} |m^{-2}F_n^{\Theta}(\theta_1 \dots \theta_n)|^2 \, e^{-rE} \right\}$$

$$r = mR \qquad E = \sum_{i=1}^{n} \cosh\theta_i$$

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
   - Zamolodchikov's c-theorem
   - The Ising and tricritical Ising steps
   - The first two steps
   - A first look into relocalisation
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Sinh-Gordon form factors

Form factors of $e^{\kappa b\phi}$ (A. Koubek & G. Mussardo, 1993)

$$F_n^{(\kappa)}(\theta_1,\theta_2,\ldots,\theta_n) = \mathcal{G}_\kappa \left(\frac{4\sin\gamma}{\mathcal{N}}\right)^{n/2} \frac{Q_n^{(\kappa)}(x_1,\ldots,x_n)}{\prod_{r<s}(x_r+x_s)} \prod_{r<s} f(\theta_r-\theta_s)$$

$$x_r = e^{\theta_r} \quad , \quad \mathcal{G}_\kappa = \langle 0|e^{\kappa b\Phi}|0\rangle$$

$$f(\theta) = \mathcal{N}\exp\left(8\int_0^\infty \frac{dt}{t}\sin^2\frac{t(i\pi-\theta)}{2\pi}\frac{\sinh\left(\frac{\gamma}{2\pi}t\right)\sinh\left(\frac{\pi-\gamma}{2\pi}t\right)\sinh\frac{t}{2}}{\sinh^2 t}\right)$$

$$\mathcal{N} = \exp\left(-4\int_0^\infty \frac{dt}{t}\frac{\sinh\left(\frac{\gamma}{2\pi}t\right)\sinh\left(\frac{\pi-\gamma}{2\pi}t\right)\sinh\frac{t}{2}}{\sinh^2 t}\right)$$

$$Q_1^{(\kappa)} = [\kappa] \qquad Q_n^{(\kappa)} = [\kappa]\det M^{(n)}(k) \qquad n>1$$

$$M_{rs}^{(n)}(\kappa) = [r-s+\kappa]\sigma_{2r-s}^{(n)}(x_1,\ldots,x_n) \quad r,s=1,\ldots,n-1$$

$$[a] = \frac{\sin a\gamma}{\sin\gamma}$$

$$\prod_{r=1}^{n}(x+x_r) = \sum_{l=1}^{n} x^{n-l}\sigma_l^{(n)}(x_1,\ldots,x_n)$$

Form factors of Θ

Form factors of Θ (Fring, Mussardo & Simonetti, 1992)

$$F_n^{\Theta}(\theta_1,\ldots,\theta_n) = \begin{cases} \frac{2\pi m^2}{F_2^{(1)}(i\pi,0)} F_n^{(1)}(\theta_1,\ldots,\theta_n) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Cluster property (Delfino, Simonetti & Cardy, 1996)

$$F_{n+l}^{(\kappa)}(\theta_1+\Lambda,\ldots,\theta_n+\Lambda,\theta_1',\ldots,\theta_l') = \frac{1}{\mathcal{G}_k} F_n^{(\kappa)}(\theta_1,\ldots,\theta_n) F_l^{(\kappa)}(\theta_1',\ldots,\theta_l') + O\left(e^{-\Lambda}\right)$$

when $\Lambda \to \infty$

$\Downarrow$ provided both n and l are even

$$F_{n+l}^{\Theta}(\theta_1+\Lambda,\ldots,\theta_n+\Lambda,\theta_1',\ldots,\theta_l') = \frac{2\sin\gamma}{\pi m^2} F_n^{\Theta}(\theta_1,\ldots,\theta_n) F_l^{\Theta}(\theta_1',\ldots,\theta_l') + O\left(e^{-\Lambda}\right)$$

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
   - Zamolodchikov's c-theorem
   - The Ising and tricritical Ising steps
   - The first two steps
   - A first look into relocalisation
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

The Ising step

Limit: $r =$ fixed while $\vartheta_0 \to \infty$ is taken:

$$F_2^{\Theta}(\theta_1, \theta_2) = -2\pi i \sinh \frac{\theta_1 - \theta_2}{2}$$

$$F_n^{\Theta}(\theta_1, \ldots, \theta_n) = 0 \quad \text{for} \quad n \neq 2$$

Apply c-sum rule

$$c(r) = 3 \sum_{n=0}^{\infty} \int \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n n!} \times \left\{ \frac{6 + 6rE + 3r^2E^2 + r^3E^3}{2E^4} |m^{-2} F_n^{\Theta}(\theta_1 \ldots \theta_n)|^2 \, e^{-rE} \right\}$$

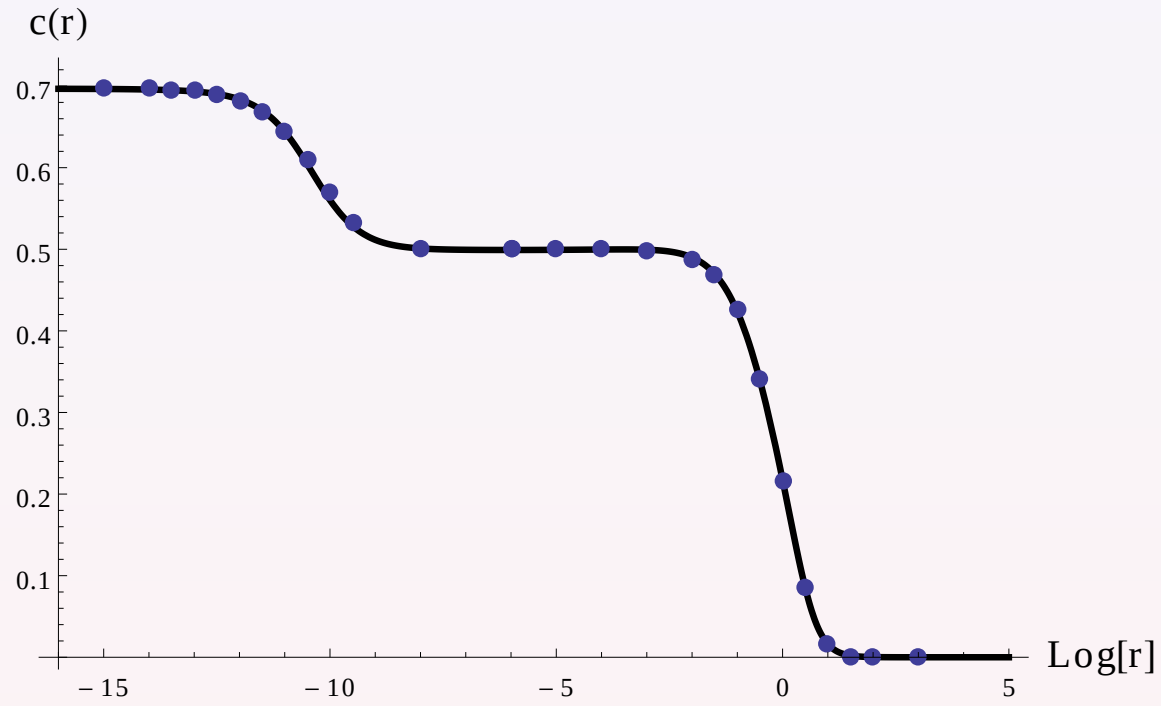
$$r = mR \qquad E = \sum_{i=1}^{n} \cosh \theta_i$$

to get (analytically)

$$\lim_{r \to 0} \left[\lim_{\vartheta_0 \to \infty} c(r, \vartheta_0) \right] = \frac{1}{2}$$

The first two steps

Do sum rule for $\vartheta_0 = 20$:



Observations:

- Staircase steps centered at $\log r = -(k-1)\frac{\vartheta_0}{2}$
- $c_1 = 1/2$ (only 2pt contributions): Ising ✔
- $c_2 = 0.69767\cdots \approx 7/10$ (2pt + 4pt): tricritical Ising ✔

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
   - Zamolodchikov's c-theorem
   - The Ising and tricritical Ising steps
   - The first two steps
   - A first look into relocalisation
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

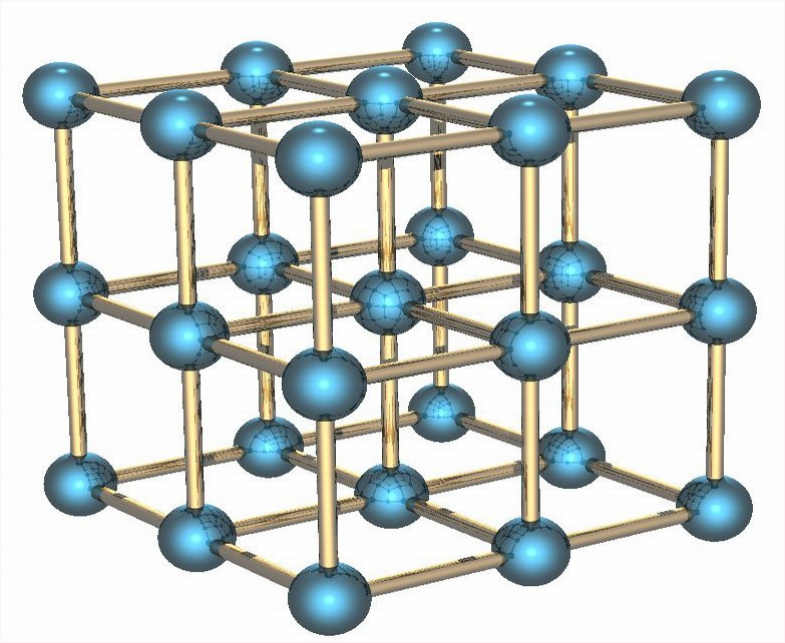
Integration cells

$$c(r) = 3\sum_{n=0}^{\infty}\int \frac{d\theta_1 \dots d\theta_n}{(2\pi)^n n!} \times \left\{ \frac{6 + 6rE + 3r^2E^2 + r^3E^3}{2E^4} |m^{-2} F_n^{\Theta}(\theta_1 \dots \theta_n)|^2 \, e^{-rE} \right\}$$

$r = mR$, $E = \sum_{i=1}^{n} \cosh\theta_i$

- e^{-rE}: restricts all rapidity integrals to $-\log(1/r) \lessapprox \theta_i \lessapprox \log(1/r)$
- steps: occur when $\log(1/r)$ passes an integer multiple of $\vartheta_0/2$

Divide rapidity space into cells (hypercubes)



$$[a_1, \dots, a_n]_\alpha := \left\{ (\theta_1, \dots, \theta_n) \in \mathbb{R}^n : a_k \frac{\vartheta_0}{2} - \frac{\alpha}{2} \le \theta_k \le a_k \frac{\vartheta_0}{2} + \frac{\alpha}{2} \right\}$$

$a_k \in \mathbb{Z}$: cell center $\quad \alpha$: cell size

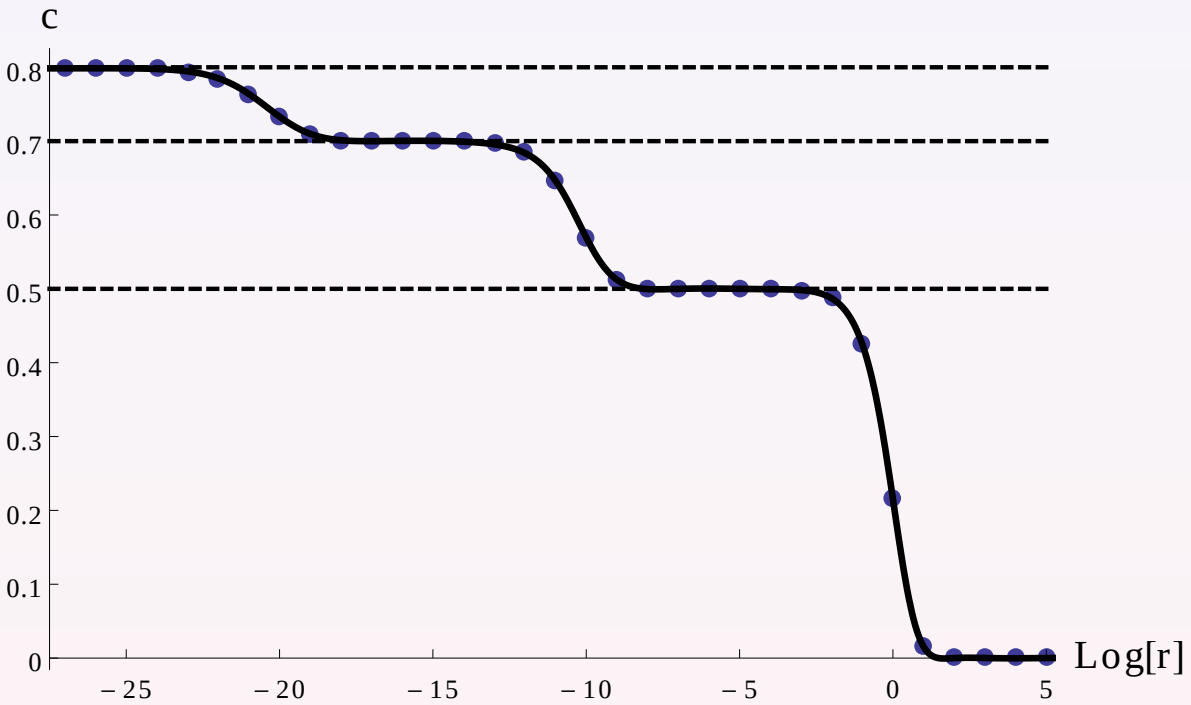
Phenomenology of relocalization

1. Each cell $[a_1, \ldots, a_n]_\alpha$ is essentially saturated by α of order 1, which does not increase with ϑ_0.
2. The contribution to the kth step starts at particle number $n = 2k$.
3. Only very few cells give non-negligible contributions; for the rest, it decays exponentially fast in ϑ_0.

Table of contributing cells in normal form
$(a_1 \leq \cdots \leq a_n\,,\ \sum_i a_i \leq 0)$

Step	2-particle	4-particle	6-particle	8-particle
1	$[0, 0]$	–	–	–
2	–	$[-1, -1, 1, 1]$	$[-1, -1, -1, -1, 1, 1]$	$[-1, -1, -1, -1, -1, -1, 1, 1]$ $[-1, -1, -1, -1, 1, 1, 1, 1]$
3	–	–	$[-2, -2, 0, 0, 2, 2]$	$[-2, -2, 0, 0, 0, 0, 2, 2]$ $[-2, -2, -2, -2, 0, 0, 2, 2]$
4	–	–	–	$[-3, -3, -1, -1, 1, 1, 3, 3]$

Three steps in the staircase



$\vartheta_0 = 20$, 2pt+4pt+6pt+8pt

Step values: $c = 0.5$

$0.6999\ldots$

$0.799\ldots$

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
   - Lukyanov representation for sinh-Gordon form factors
   - Power counting for form factors in the roaming limit
   - The c-theorem sum rule in the roaming limit
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
   - Lukyanov representation for sinh-Gordon form factors
   - Power counting for form factors in the roaming limit
   - The c-theorem sum rule in the roaming limit
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Lukyanov's representation for the form factors

$$F_n^{(\kappa)}(\theta_1,\ldots,\theta_n) = \mathcal{G}_\kappa \frac{i^n}{(\mathcal{N}\sin\gamma)^{n/2}} P_n^{(\kappa)}(\theta_1,\ldots,\theta_n) \prod_{r<s} f(\theta_r-\theta_s)$$

$$P_n^{(\kappa)}(\theta_1,\ldots,\theta_n) = \sum_{\{\alpha_j=\pm 1\}} \left[\prod_{j=1}^{n} \alpha_j e^{-i\alpha_j \kappa\gamma} \prod_{r<s} \left(1 - i\frac{\alpha_r-\alpha_s}{2}\frac{\sin\gamma}{\sinh(\theta_r-\theta_s)}\right)\right]$$

From this representation cluster property is trivial

$$F_{n+l}^{(\kappa)}(\theta_1+\Lambda,\ldots,\theta_n+\Lambda,\theta'_1,\ldots,\theta'_l)$$
$$= \frac{1}{\mathcal{G}_k} F_n^{(\kappa)}(\theta_1,\ldots,\theta_n) F_l^{(\kappa)}(\theta'_1,\ldots,\theta'_l) + O\left(e^{-\Lambda}\right)$$
$$\text{when } \Lambda\to\infty$$

Relation to determinant representation

$$P_n^{(\kappa)}(\theta_1,\ldots,\theta_n) = (-2i\sin\gamma)^n \frac{Q_n^{(\kappa)}(e^{\theta_1},\ldots,e^{\theta_n})}{\prod_{r<s}(e^{\theta_r}+e^{\theta_s})}$$

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
   - Lukyanov representation for sinh-Gordon form factors
   - Power counting for form factors in the roaming limit
   - The c-theorem sum rule in the roaming limit
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Roaming limit of minimal form factor

$$f(\theta) = \mathcal{N} R(\theta)$$

$$R(\theta) = \exp\left[8\int_0^\infty \frac{dt}{t}\sin^2\left(\frac{t(i\pi-\theta)}{2\pi}\right)\frac{\sinh\frac{t\gamma}{2\pi}\sinh(1-\frac{\gamma}{\pi})\frac{t}{2}\sinh\frac{t}{2}}{\sinh^2 t}\right]$$

$$R(\theta+k\vartheta_0) \underset{\vartheta_0\to 0}{\longrightarrow} \begin{cases} -i\sinh\theta/2 & k=0 \\ e^{\vartheta_0/2}\rho(\theta) & k=+1 \\ e^{\vartheta_0/2}\bar{\rho}(\theta) & k=-1 \\ \frac{1}{2}e^{\vartheta_0/2} & |k|>1 \end{cases}$$

where

$$\rho(\theta) = \frac{1-i}{4}2^{1/4}e^{-K/\pi}e^{\theta/4}\exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sin^2\left(\frac{t(i\pi-\theta)}{2\pi}\right)}{\sinh t\cosh\frac{t}{2}}\right)$$

$$\bar{\rho}(\theta) = \frac{1+i}{4}2^{1/4}e^{-K/\pi}e^{-\theta/4}\exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sin^2\left(\frac{t(i\pi-\theta)}{2\pi}\right)}{\sinh t\cosh\frac{t}{2}}\right)$$

$$K = 0.915966\ldots \text{ (Catalan's constant)}$$

$$\rho(\infty) = \bar{\rho}(-\infty) = \frac{1}{2} \quad \rho(-\infty) = \bar{\rho}(\infty) = 0$$

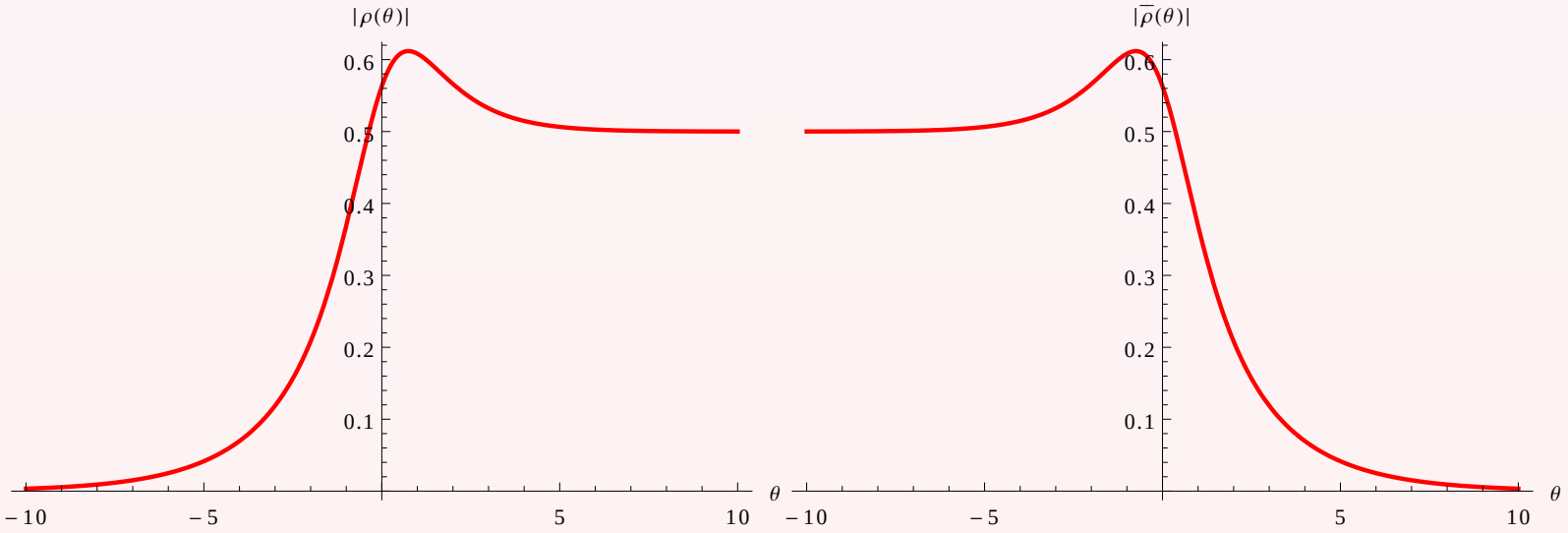
$$\mathcal{N} \sim 2e^{-\vartheta_0/2}$$

Roaming limit of minimal form factor

The universal minimal roaming form factor

$$f(\theta + k\vartheta_0) \underset{\vartheta_0 \to 0}{\longrightarrow} \begin{cases} -2ie^{-\vartheta_0/2} \sinh\frac{\theta}{2} & k = 0 \\ 2\rho(\theta) & k = +1 \\ 2\bar{\rho}(\theta) & k = -1 \\ 1 & |k| > 1 \end{cases}$$

$$\rho(\theta) = \frac{1-i}{4} 2^{1/4} e^{-K/\pi} e^{\theta/4} \exp\left(-\int_0^\infty \frac{dt}{t} \frac{\sin^2\left(\frac{t(i\pi-\theta)}{2\pi}\right)}{\sinh t \cosh\frac{t}{2}} \right)$$



Naive power counting

Choose a cell of normal form:$[a_1, \ldots, a_n]$ with $a_1 \leq \cdots \leq a_n$ and define $\vec{t}$ as

$$t_i = (a_i - a_1)/2 + 1 \quad i = 1, \ldots, n$$

Integrality assumption (I): $t_i \in \mathbb{N}$ (holds empirically).
Power counting for the form factor in the cell $[a_1, \ldots, a_n]$

$$F_n^{\Theta}(\theta_1, \ldots, \theta_n) \sim e^{\omega(\vec{t})\vartheta_0} \quad \text{while } \theta_i = \theta_i' + t_i\vartheta_0 \quad \text{with } \theta_i' \text{ finite}$$

$$\text{or} \quad \omega(\vec{t}) := \lim_{\vartheta_0 \to \infty} \frac{1}{\vartheta_0} \log F_n^{\Theta}\left(\theta_1' + t_1\vartheta_0, \ldots, \theta_n' + t_n\vartheta_0\right)$$

Putting together all

$$\omega(\vec{t}) = -1 - \tfrac{n}{4} + \omega_P(\vec{t}) + \omega_F(\vec{t})$$

-1 : $F_2(0, i\pi)$ normalization $\quad \omega_P(\vec{t})$: $P_n^{(1)}$ (Lukyanov rep)

$\omega_F(\vec{t}) = \sum_{<i,j>} \min\left(0, \frac{|t_{ij}|-1}{2}\right)$: minimal form factors

Upper estimate for ω_P

$$P_n^{(1)} \sim \sum_{\vec{\alpha}} \left(\prod_{j=1}^{n} \alpha_j e^{-\alpha_j(\vartheta_0 + i\pi/2)} \right) \prod_{k<j} \left(1 - i \frac{\alpha_k - \alpha_j}{2} \frac{\cosh \vartheta_0}{\sinh(\theta_k - \theta_j)} \right)$$

For each $\vec{\alpha}$ we can define an exponent $\omega_P(\vec{\alpha}, \vec{t}\,)$, then

$$\omega_P(\vec{t}\,) \leq \tilde{\omega}_P(\vec{t}\,) = \max_{\vec{\alpha}} \omega_P(\vec{\alpha}, \vec{t}\,)$$

Writing $\vec{t}$ in a block form

$$\vec{t} \;=\; (\underbrace{v_1 \dots v_1}_{p_1}, \underbrace{v_2 \dots v_2}_{p_2}, \dots, \underbrace{v_N \dots v_N}_{p_N})$$

$$v_N = \max \vec{t}, \; v_1 < v_2 < \dots < v_N \qquad \sum_{k=1}^{N} p_k = n$$

the result is

$$\tilde{\omega}_P(\vec{t}\,) = n + \sum_{k=1}^{N} \left[\left(\frac{p_k - 2}{2} \right)^2 - \frac{\delta_k^2}{4} \right] \qquad \delta_k = \begin{cases} 0 & p_k \text{ even} \\ 1 & p_k \text{ odd} \end{cases}$$

Upper estimate for $\omega(\vec{t}\,)$

Adding to $\tilde{\omega}_P(\vec{t}\,)$ the minimal form factor contribution $\omega_F(\vec{t}\,)$:

$$\begin{aligned}\omega(\vec{t}\,) \le \tilde{\omega}(\vec{t}\,) &= -1 + \frac{3n}{4} + \sum_{k=1}^{N}\left(\frac{p_k-2}{2}\right)^2 - \sum_{k=1}^{N}\frac{\delta_k^2}{4} \\ &\quad + \sum_{\langle i,j\rangle}\min\left(0, \frac{|t_{ij}|-1}{2}\right) \\ &= -1 + N - \sum_{k=1}^{N}\frac{\delta_k^2}{4} \qquad \text{using: } \sum_{k=1}^{N} p_k = n\end{aligned}$$

So dominant cells must have blocks of even length and the largest number of blocks possible, i.e. the smallest step between blocks:

$$\vec{t} = (\underbrace{v_1 \ldots v_1}_{p_1}, \underbrace{v_2 \ldots v_2}_{p_2}, \ldots, \underbrace{v_N \ldots v_N}_{p_N})$$

all p_i even and $v_{k+1} - v_k = 1$

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
   - Lukyanov representation for sinh-Gordon form factors
   - Power counting for form factors in the roaming limit
   - The c-theorem sum rule in the roaming limit
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Scaling the c-theorem contributions by cell

Cell $[a_1, \ldots, a_n]$ of normal form $(a_1 \leq \cdots \leq a_n)$: we get $\vec{t}$ as $t_i = (a_i - a_1)/2 + 1$. We have shown that

$$\omega(\vec{t}\,) \leq \tilde{\omega}(\vec{t}\,) = N - 1$$

The contribution of the cell to $c(r)$ is

$$\int_{[\vec{a}]} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n (n!)} \frac{6 + 6rE + 3r^2E^2 + r^3E^3}{2E^4} \left| m^{-2} F_n^{\Theta}(\theta_1 \ldots \theta_n) \right|^2 e^{-rE}$$

and its coordinates must satisfy

$$\log r \leq a_1 \frac{\vartheta_0}{2} \leq \cdots \leq a_n \frac{\vartheta_0}{2} \leq -\log r \qquad \Rightarrow \qquad rE \lesssim O(1)$$

so the scaling in ϑ_0 is given by

$$\int_{[\vec{a}]} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n (n!)} \frac{\left| m^{-2} F_n^{\Theta}(\theta_1 \ldots \theta_n) \right|^2}{2E^4} \propto e^{\zeta(\vec{a})\vartheta_0}$$

Length of $\vec{t}$:

$$D(\vec{t}\,) = t_n - t_1 \qquad \text{so} \qquad a_n - a_1 = 2(D(\vec{t}\,) - 1)$$

Dominant cells

Denoting $k = D(\vec{t}\,)$, cells giving the highest possible contribution at volume r have the form

$$[\vec{a}] \;=\; [\underbrace{-k+1,\ldots,-k+1}_{p_1},\underbrace{-k+1,\ldots,-k+1}_{p_2},\ldots,\underbrace{k-1,\ldots,k-1}_{p_k}] \quad \checkmark$$

p_l : positive even number and $\qquad (k-1)\vartheta_0/2 \leq -\log r$

For such a cell $N = k$. Recalling

$$\int_{[\vec{a}]} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n (n!)} \, \frac{|m^{-2} F_n^{\Theta}(\theta_1 \ldots \theta_n)|^2}{2E^4} \propto e^{\zeta(\vec{a})\vartheta_0}$$

$$E^4 \propto \exp\{2(k-1)\vartheta_0\}$$

$$\Downarrow$$

$$\zeta(\vec{a}) = 2\omega(\vec{t}\,) - 2(k-1) < 2(N-k) = 0$$

Dominant cells scale to $\leq O(1) \;\Rightarrow$ subdominants have no chance!

Remaining issue: do all dominant cells really contribute?

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
   - The scaling anomaly
   - The integrality assumption
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
   - The scaling anomaly
   - The integrality assumption
5. Massless form factors: a reconstruction
6. Conclusion and outlook

The scaling anomaly

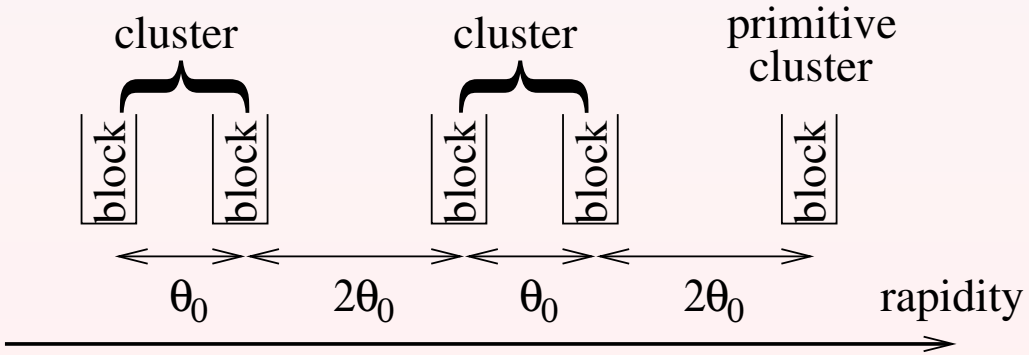
So far: only upper bound on cell contributions.

Scaling anomaly:

$$\begin{aligned} \omega(\vec{t}\,) &= \tilde{\omega}(\vec{t}\,) + \Delta\omega(\vec{t}\,) \\ & \quad \Delta\omega(\vec{t}\,) = \omega_P(\vec{t}\,) - \tilde{\omega}_P(\vec{t}\,) \leq 0 \\ \vec{t} &= (\underbrace{v_1 \ldots v_1}_{p_1}, \underbrace{v_2 \ldots v_2}_{p_2}, \ldots, \underbrace{v_N \ldots v_N}_{p_N}) \\ & \quad 1 = v_1 < v_2 < \cdots < v_N \leq n \end{aligned}$$

Classification of index pairs $\langle j, k \rangle$:

1. *internal* if $|t_{jk}| = 0$; $a_j = a_k$, i.e. from inside the same block;
2. *strongly linked* if $|t_{jk}| = 1$;
3. *weakly linked* if $|t_{jk}| \geq 2$.



The scaling anomaly

Result:

$$\Delta\omega(\vec{t}\,) = -\sum_{\substack{\mathcal{C}:\ \text{primitive} \\ \text{cluster}}} \frac{\text{length}(\mathcal{C}) - 2}{2} \,.$$

Dominant cells:

$$\vec{t} = (\underbrace{1\ldots1}_{p_1}, \underbrace{2\ldots2}_{p_2}, \ldots, \underbrace{k\ldots k}_{p_k})$$

$k \geq 2$: $\vec{t}$ is a single non-primitive cluster $\Rightarrow$ zero anomaly! ✔
$k = 1$: single primitive cluster

$$\begin{aligned} \vec{t} &= (\underbrace{1\ldots1}_{p}) \\ \zeta(\vec{t}\,) &= 2\omega(\vec{t}\,) - 2(k-1) = 2\Delta\omega(\vec{t}\,) = -p + 2 \end{aligned}$$

i.e. only 2pt ($p = 2$) contributions for Ising step! ✔

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
   - The scaling anomaly
   - The integrality assumption
5. Massless form factors: a reconstruction
6. Conclusion and outlook

The integrality assumption

Remaining assumption: coordinates of cells are integers

$$\begin{aligned}\vec{t} &= (\underbrace{v_1 \dots v_1}_{p_1}, \underbrace{v_2 \dots v_2}_{p_2}, \dots, \underbrace{v_N \dots v_N}_{p_N}) \\ & v_1 < v_2 < \cdots < v_N \\ & t_{ij} = t_i - t_j \qquad v_{ij} = v_i - v_j \\ D(\vec{t}) &= v_N - v_1\end{aligned}$$

Need: scaling exponent for the $\vec{\alpha}$ term

$$F_n^{(\kappa)}(\theta_1, \dots, \theta_n) = \mathcal{G}_\kappa \frac{i^n}{(\mathcal{N} \sin\gamma)^{n/2}} P_n^{(\kappa)}(\theta_1, \dots, \theta_n) \prod_{r<s} f(\theta_r - \theta_s)$$

$$P_n^{(\kappa)}(\theta_1, \dots, \theta_n) = \sum_{\{\alpha_j = \pm 1\}} \left[\prod_{j=1}^{n} \alpha_j e^{-i\alpha_j \kappa\gamma} \prod_{r<s} \left(1 - i \frac{\alpha_r - \alpha_s}{2} \frac{\sin\gamma}{\sinh(\theta_r - \theta_s)} \right) \right]$$

with

$$\begin{aligned}\vec{\alpha} &= (\underbrace{\pm 1 \cdots \pm 1}_{p_1}, \underbrace{\pm 1 \cdots \pm 1}_{p_2}, \dots, \underbrace{\pm 1 \cdots \pm 1}_{p_N}) \\ & r_k = \#\text{ of } +1\text{'s in block } k, \qquad k = 1, \dots, N \\ & s_k = p_k - 2r_k\end{aligned}$$

The integrality assumption

Scaling exponent for the $\vec{\alpha}$ term

$$\omega(\vec{\alpha},\vec{t}\,) \;=\; -1+N-\frac{1}{4}\sum_{k=1}^{N}(s_k-2)^2-\sum_{<i,j>}\frac{A_{ij}}{2}s_i s_j$$

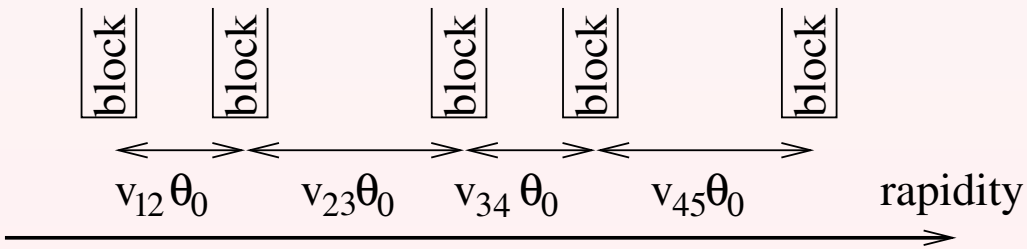
$$A_{ij}=\frac{1-|v_{ij}|}{2}\theta(1-|v_{ij}|)$$

Optimizing ζ

$$\int_{[\vec{a}]}\frac{d\theta_1\ldots d\theta_n}{(2\pi)^n(n!)}\,\frac{|m^{-2}F_n^{\Theta}(\theta_1\ldots\theta_n)|^2}{2E^4}\propto e^{\zeta(\vec{a})\vartheta_0}$$

we need to lower E i.e. $D(\vec{t})=\max(\vec{t})-\min(\vec{t})$, leading to

$$|v_{i+1}-v_i|\leq 1$$



The integrality assumption

Supposing integrality:

$$\begin{aligned} v_{i+1} - v_i &= 1 \\ &\Downarrow \\ A_{ij} &= \frac{1-|v_{ij}|}{2}\theta(1-|v_{ij}|) = 0 \\ \omega(\vec{\alpha},\vec{t}\,) &= -1+N-\frac{1}{4}\sum_{k=1}^{N}(s_k-2)^2 \end{aligned}$$

Now $s_k = p_k - 2r_k$

$s_k \bmod 2 = p_k \bmod 2 = \delta_k$

so

$$\max_{\vec{\alpha}} \omega(\vec{\alpha},\vec{t}\,) = -1+N-\sum_{k=1}^{N}\frac{\delta_k^2}{4}$$

Relaxing integrality:

$$\begin{aligned} A_{ij} &\neq 0 \\ \max_{\vec{\alpha}} \omega(\vec{\alpha},\vec{t}\,) &= -1+N-\sum_{k=1}^{N}\frac{\delta_k^2}{4} + \mathcal{C}(A) \end{aligned}$$

Now: $\mathcal{C}(A) < 0$ to first order in A

Go away slightly from integrality: it drowns!

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Reconstructing massless form factors

In the c-theorem sum rule

$$c(r) = 3\sum_{n=0}^{\infty}\int \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n n!} \times\left\{\frac{6+6rE+3r^2E^2+r^3E^3}{2E^4}|m^{-2}F_n^{\Theta}(\theta_1\ldots\theta_n)|^2\, e^{-rE}\right\}$$

$$r = mR \qquad E = \sum_{i=1}^{n}\cosh\theta_i$$

write

$$\int = \sum_{[\vec{a}]}\int_{[\vec{a}]}$$

In cell $[a_1,\ldots,a_n]$ parameterise by integration variables β_i

$$\theta_i = \beta_i + a_i\frac{\vartheta_0}{2} \qquad i = 1,\ldots,n$$

and rescale to localise the integral to step k:

$$r = \tilde{r}e^{-(k-1)\vartheta_0/2}$$

Reconstructing massless form factors

Result: the c-function for the flow $\mathcal{M}_{k+2} \longrightarrow \mathcal{M}_{k+1}$

$$c_k(r) = \lim_{\vartheta_0 \to \infty} \sum_{n=0}^{\infty} \sum_{\substack{[a_1,\ldots,a_n] \\ a_n = k-1}} \int_{[a_1,\ldots,a_n]} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n (n!)} \left[\frac{6 + 6rE + 3r^2E^2 + r^3E^3}{2E^4} \right.$$

$$\left. \times e^{-rE} |m^{-2} F_n^{\Theta}(\beta_1 + a_1\vartheta_0/2, \ldots, \beta_n + a_n\vartheta_0/2)|^2 \right]$$

$$E = \sum_{i=1}^{n} \cosh(\beta_i + a_i\vartheta_0/2)$$

Write

$$[a_1, \ldots, a_n] = [\underbrace{-k+1, \ldots, -k+1}_{2p_{-k+1}}, \underbrace{-k+3, \ldots, -k+3}_{2p_{-k+3}}, \ldots, \underbrace{k-1, \ldots, k-1}_{2p_{k-1}}]$$

$$\{\beta_1, \ldots \beta_n\} = \{\beta_1^{(-k+1)}, \ldots, \beta_{2p_{-k+1}}^{(-k+1)}, \beta_1^{(-k+3)}, \ldots, \beta_{2p_{-k+3}}^{(-k+3)}, \ldots, \beta_1^{(k-1)}, \ldots, \beta_{2p_{k-1}}^{(k-1)}\}$$

Reconstructing massless form factors

c-function for the flow $\mathcal{M}_{k+2} \longrightarrow \mathcal{M}_{k+1}$

$$c_k(r) = \sum_{n=0}^{\infty} \sum_{\substack{\{p_i\} \\ 2\sum_i p_i = n}} \frac{1}{n!} \int \left(\prod_{i=-k+1}^{k-1} \prod_{j_i=1}^{2p_i} \frac{d\beta_{j_i}^{(p_i)}}{2\pi} \right)$$

$$\left[\frac{6 + 6\tilde{r}\tilde{E} + 3\tilde{r}^2\tilde{E}^2 + \tilde{r}^3\tilde{E}^3}{2\tilde{E}^4} e^{-\tilde{r}\tilde{E}} \right.$$

$$\left. \times \left| \mathcal{F}^{\Theta}_{\{p_i\}} \left(\beta_1^{(-k+1)}, \ldots, \beta_{2p_{-k+1}}^{(-k+1)}, \ldots, \beta_1^{(k-1)}, \ldots, \beta_{2p_{k-1}}^{(k-1)} \right) \right|^2 \right]$$

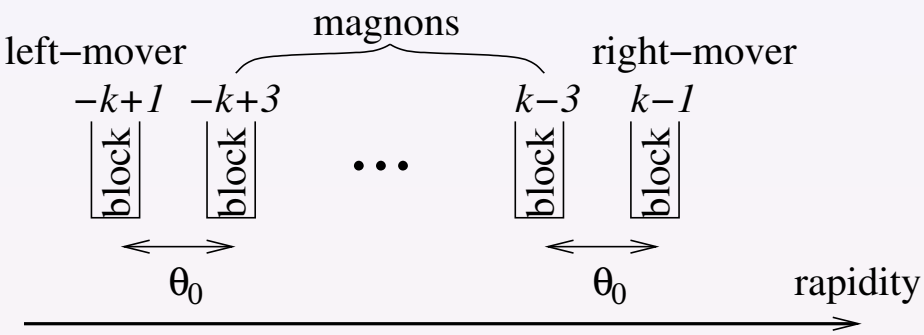
$$\tilde{E} = \sum_{i=1}^{2p_{-k+1}} e^{-\beta_i^{(p_{-k+1})}} + \sum_{i=1}^{2p_{k-1}} e^{\beta_i^{(p_{k-1})}}$$

where $\{p_i\}$ is a shorthand for $\{p_{-k+1}, p_{-k+3}, \ldots, p_{k-1}\}$ and

$$\mathcal{F}^{\Theta}_{\{p_i\}} \left(\beta_1^{(-k+1)}, \ldots, \beta_{2p_{-k+1}}^{(-k+1)}, \ldots, \beta_1^{(k-1)}, \ldots, \beta_{2p_{k-1}}^{(k-1)} \right) =$$

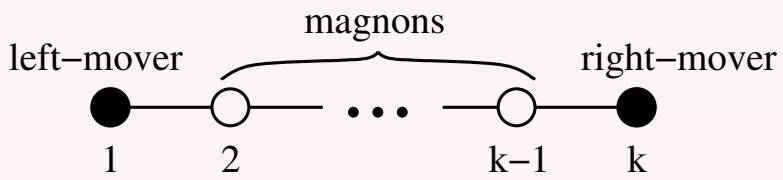
$$\lim_{\vartheta_0 \to \infty} m^{-2} e^{-(k-1)\vartheta_0} F_n^{\Theta}(\beta_1 + a_1\vartheta_0/2, \ldots, \beta_n + a_n\vartheta_0/2)$$

Magnonic interpretation



$$\tilde{E} = \sum_{i=1}^{2p_{-k+1}} e^{-\beta_i^{(p_{-k+1})}} + \sum_{i=1}^{2p_{k-1}} e^{\beta_i^{(p_{k-1})}}$$

Identical to the Dynkin diagram of the RSOS TBA!



$$\begin{aligned}
\epsilon_1(\theta) &= \frac{\tilde{r}}{2}e^{\theta} - \int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\log\left(1+e^{-\epsilon_2(\theta')}\right) \\
\epsilon_j(\theta) &= -\int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\left\{\log\left(1+e^{-\epsilon_{j-1}(\theta')}\right)+\log\left(1+e^{-\epsilon_{j+1}(\theta')}\right)\right\} \\
\epsilon_k(\theta) &= \frac{\tilde{r}}{2}e^{-\theta} - \int \frac{d\theta'}{2\pi}\frac{1}{\cosh(\theta-\theta')}\log\left(1+e^{-\epsilon_{k-1}(\theta')}\right)
\end{aligned}$$

Magnons and kinks

The actual spectrum along the flow: left/right-moving kinks

$$|K^L_{a_0 a_1}(\theta_1)\dots K^L_{a_{n-1}a_n}(\theta_n)K^R_{a_n a_{n+1}}(\theta'_1)\dots K^R_{a_{n+m-1}a_{n+m}}(\theta'_m)\rangle$$

$$E = M\sum_{l=1}^{n} e^{-\theta_l} + M\sum_{l=n+1}^{n+m} e^{\theta_l} \qquad P = -M\sum_{l=1}^{n} e^{-\theta_l} + M\sum_{l=n+1}^{n+m} e^{\theta_l}$$

RSOS sequence: internal states of the kinks

$$a_0 a_1 \dots a_{n+m} \qquad a_l \in \{0,\dots,k\}$$

$$\text{adjacency rules} \quad : \quad |a_l - a_{l+1}| = 1$$

Scattering: massless factorized S-matrix
(Fendley, Saleur & Zamolodchikov, 1993)
TBA: magnons parameterise the internal states.

$$\text{RSOS basis} \leftrightarrow \text{magnonic basis}$$

$k = 1$: no magnons, FFs are known
(Delfino, Mussardo & Simonetti, 1995)

Checked: they agree with our reconstruction! ✔

Outline

1. Introduction: Zamolodchikov's roaming flows
2. C-theorem for roaming flows
3. Relocalisation: analytical approach
4. A deeper look at relocalisation
5. Massless form factors: a reconstruction
6. Conclusion and outlook

Conclusion and outlook

Summary of results

1. Zamolodchikov's sinh-Gordon TBA roaming flows: reproduced from the c-theorem.
2. Interesting phenomenon: form factor relocalization (almost) proven as a theorem!
3. Construction of form factors along massless flows with non-diagonal scattering.

Open problems

1. Construct form factors for other operators: work in progress.
2. Implement for other roaming families, plus integrable models with resonances (homogeneous sine-Gordon models).
3. Dynkin form factors – analogue of Dynkin TBA?
    1. Diagrammatic construction of FFs seems possible.
    2. Avoids solving form factor bootstrap (very tricky for non-diagonal scattering)