SOLITON SOLUTIONS

OF CLASSICAL EQUATIONS OF MOTION

OF THE MODIFIED YANG-MILLS THEORY.

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Infrared properties of the Yang-Mills field remain rather obscure. Analytic proof of the confinement of color still is absent. Numerical simulations indicate on the existence of color cofinement, but they usually rely on the existence of soliton- like excitations which are absent in the usual Yang-Mills theory.

Perturbative scattering matrix does not exist in the Yang-Mills theory, and the only sensible objects are the correlation functions of the gauge invariant operators.

In the papers (A.A.Slavnov, JHEP 0808 (2008) 047, A.A.Slavnov, Theoretical and Mathematical Physics 161(2009)204,A.Quadri, A.A.Slavnov JHEP 07(2010) 087), a new formulation of nonabelian gauge theories, applicable not only in perturbation theory but also beyond it, was proposed. The usual arguments forbidding the existence of solitons in the Yang-Mills theory are not applicable to this formulation. In this talk I am going to show that the solitons of the t'Hooft-Polyakov magnetic monopole type indeed are possible in the Yang-Mils theory. The material of this talk is published in hep-th arXiv 1406.7724(2014).

We start with the non-Abelian model, described by the Lagrangian

$$L = -\frac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu} + \frac{1}{2}D_\mu\varphi^i D_\mu\varphi^i - \frac{1}{2}D_\mu\chi^i D_\mu\chi^i + iD_\mu b^i D_\mu e^i \qquad (1)$$

For simplicity we consider the case of $SU(2)$ gauge group. The scalar fields (φ, χ) are commuting, e, b are anticommuting. All these fields belong to the adjoint representation of the gauge group. In this equation D_μ is the usual covariant derivative, hence the Lagrangian (1) is gauge invariant. Note the minus sign before the terms, describing the field χ. Because of the sign they possess the negative energy.

We assume (and check later) that the fields, entering the Lagrangian (1) have the following asymptotic behaviour

$$|\varphi| \to |\frac{m}{g}|; \quad |\chi| \to |\frac{m\alpha}{g}|; \quad r = |\mathbf{x}|; \quad r \to \infty \tag{2}$$

The parameter $\alpha \to 1$ when $g \to 0$ sufficiently fast. For example we may take

$$\alpha = \frac{g^{-n} - g^{n}}{g^{-n} + g^{n}} = 1 - g^{2n} + \dots \tag{3}$$

So that $1 - \alpha = O(g^{2n})$, and choosing n big enough we get in a formal perturbation theory the results, coinciding with the standard Yang-Mills theory to arbitrary order in g. In the eq.(2) m is a constant having dimension of mass.

Speaking about formal perturbation theory, we assume formal series in the coupling constant, no matter is it convergent or not. This is the usual notion for quantum field theory. If the coupling constant g is small, like in quantum electrodynamics it means that the usual relations between the elements of the scattering matrix (unitarity, causality e.t.c.) are approximately fulfilled in a given order of formal perturbation theory. But in the theories like quantum chromodynamics (QCD) the coupling constant is not small and even the separate terms in the formal perturbation series may not exist due to infrared divergencies. Nevertheless one usually insists on the formal relations like unitarity and causality in QCD. This point of view is supported by the fact that correlators of the gauge invariant operators as a rule have no infrared singularities. One may hope that for a proper choice of asymptotic states these problems may be avoided.

In a topologically trivial sector corresponding to the perturbation theory we can choose the direction, where the asymptotic does not vanish, as the third axis in the charge space. Making the shifts of the fields preserving manifest Lorentz invariance

$$\varphi^i = \tilde{\varphi}^i + \delta^{i3} m g^{-1}; \quad \chi^i = \tilde{\chi}^i - \delta^{i3} m \alpha g^{-1} \tag{4}$$

we get the Lagrangian with fields φ, χ going to zero at $r \to \infty$. This is necessary to develop a perturbation expansion near the vacuum state.

We want to prove that the scattering matrix obtained after this shift in the framework of perturbation theory coincides with the usual scattering matrix in the Yang-Mills theory. If $\alpha \neq 1$, we may speak about the coincidence of the scattering matrices up to arbitrary order in formal perturbation theory. For any term in the perturbative expansion of the scattering matrix we may choose n big enough to get the complete agreement up to this term. In the Yang-Mills theory the scattering matrix does not exist due to infrared divergencies, but one can nevertheless speak about the absence of transitions between the states including only physical excitations and the states including some unphysical ones. Physical excitations in both theories are three dimensionally transversal components of the Yang-Mills field.

For $\alpha \neq 1$ the theory we consider is not the standard Yang-Mills theory. But it is gauge invariant for any α, and the values of observables calculated using a formal perturbation theory in the coupling constant coincide to any order in the coupling constant with the values, calculated in the usual Yang-Mills theory. Moreover this formulation may be used beyond perturbation theory and does not suffer from the Gribov ambiguity (A.A.S.). If the coupling constant g is very small, and the limit at $\alpha \to 1$ exists for observables, as it happens in the electro-weak models based on the Higgs-Brout-Englert mechanism, no solitons are required. But in quantum chromodynamics this limit does not exist for the on shell scattering matrix elements due to infrared divergencies, and the coupling constant g is not small, so that α may differ considerably from unity. In this case as we shall show soliton excitations may arise.

In perturbation theory there is no Gribov ambiguity, so we can choose the gauge $\partial_\mu A^i_\mu = 0$ introducing also the Faddeev-Popov ghosts $\bar{c}^i, c^i$.

The scattering matrix for $\alpha = 1$ may be presented by the path integral

$$S = \int d\mu \{ i [\int d^4x (-\frac{1}{4} F^i_{\mu\nu} F^i_{\mu\nu} + D_\mu \tilde{\varphi}^i_+ D_\mu \tilde{\varphi}^i_- +$$
$$+ \lambda^i \partial_\mu A^i_\mu + i \partial_\mu \bar{c}^i D_\mu c^i + i D_\mu \tilde{b}^i D_\mu \tilde{e}^i + m D_\mu \tilde{\varphi}^i_+ \varepsilon^{ij3} A^j_\mu)] \} \qquad (5)$$

The results obtained with the help of the shifted Lagrangian coincide with the usual ones at least to the order $2n$. As the number n is arbitrary that means the formal perturbation expansion obtained in this way coincides with the usual one. Being interested in the perturbative results we may put $\alpha = 1$, as $\alpha = 1 - O(g^{2n})$, where n is an arbitrary number. Clearly in this case no mass term for the Yang-Mills field is generated, as due to the opposite signs of the terms depending on φ and χ their contributions to the mass of the Yang-Mills field cancel.

For $\alpha = 1$ the action is also invariant with respect to supersymmetry transformation

$$\delta\tilde{\varphi}^i_- = i\tilde{b}^i\epsilon; \quad \delta\tilde{e}^i = \tilde{\varphi}^i_+\epsilon; \quad \delta\tilde{b}^i = \delta\tilde{\varphi}^i_+ = 0. \tag{6}$$

It is easy to see that these transformations are nilpotent

$$\delta^2\tilde{\varphi}^i_- = 0; \quad \delta^2\tilde{e}^i = 0 \tag{7}$$

This invariance provides the decoupling of excitations corresponding to the fields $\tilde{\varphi}_\pm, \tilde{b}, \tilde{e}$.

According to the Noether theorem these symmetries generate conserved charges Q_B, Q_S. The corresponding asymptotic conserved charges are denoted as Q_B^0 and Q_S^0 and the asymptotic states may be chosen to satisfy the equations

$$Q_B^0|\psi>_{ph} = 0; \quad Q_S^0|\psi>_{ph} = 0; \quad [Q_B^0, Q_S^0]_+ = 0 \tag{8}$$

Q_B^0 and Q_S^0 are the asymptotic charges equal to

$$Q_B^0 = \int d^3x[(\partial_i A_0 - \partial_0 A_i)^j \partial_i c^j - \lambda^j \partial_0 c^j] \quad (9)$$

$$Q_S^0 = \int d^3x(\partial_0 \varphi_+^i b^i - \partial_0 b^i \varphi_+^i) \quad (10)$$

The second equation (8) provides the decoupling of the excitations corresponding to the fields $\varphi_\pm, e, b$. The first equality is analogous to the corresponding equality in the BRST treatment of the standard Yang-Mills theory. It guarantees the absence of the transitions from the states containing only the excitations corresponding to the transversal components of the Yang-Mills field to the states containing longitudinal and temporal quanta.

The structure of the vectors, annihilated by the operators Q may be studied in the following way. It is possible to introduce the operators K, satisfying the following conditions

$$[\hat{K}, \hat{Q}]_+ = \hat{N} \tag{11}$$

where $\hat{N}$ is the number operator for the ghost particles. Any vector which contains at least one ghost particle may be presented in the form

$$|\chi>= \frac{[\hat{K}, \hat{Q}]_+}{N}|\chi>= \frac{\hat{K}\hat{Q}}{N}|\chi> + \frac{\hat{Q}\hat{K}}{N}|\chi> \tag{12}$$

The first term is zero for the state vectors annihilated by $\hat{Q}$. The second term has zero norm.

Any vector, satisfying eqs. (8) has a structure

$$|\psi>_{ph} = |\psi>_{tr} + |N> \tag{13}$$

where $|\psi>_{tr}$ is a vector which contains only transversal quanta and $|N>$ is a zero norm vector.

Factorising this subspace with respect to the vectors $|N>$, we get the physical space which coincide with the space of states of the Yang-Mills theory. So we proved the perturbative unitarity of the Yang-Mills scattering matrix in the space, which contains only physical excitations. The proof however was formal, as the on shell scattering matrix elements do not exist due to infrared singularities. We can however speak about nullification of the matrix elements corresponding to transitions between physical and unphysical states.

Now we are going to prove that the classical equations, corresponding to the Lagrangian(1) have soliton solutions of the t'Hooft-Polyakov type. We shall look for the classical solitons with the asymptotic for large r

$$\varphi^i \to \frac{x^i m}{rg}; \quad \chi^i \to -\frac{x^i m\alpha}{rg} \tag{14}$$

We are working with the stationary solutions in the gauge $A_0 = 0$.

The classical equations and asymptotic conditions look as follows

$$D_j F^l_{ij} + g\varepsilon^{ijm}(D_j\varphi)^l\varphi^m - g\varepsilon^{ijn}(D_j\chi)^l\chi^n = 0; \quad A^l_i \to \varepsilon^{lik}\frac{x^k}{gr^2}, r \to \infty$$
$$D_i(D_i\varphi)^n = 0; \quad \varphi^j(x) \to \frac{x^j m}{gr}, r \to \infty$$
$$D_i(D_i\chi)^n = 0; \quad \chi^j(x) \to -\frac{\alpha x^j m}{gr}, r \to \infty. \quad (15)$$

These boundary conditions provide decreasing of covariant derivatives of the fields φ, χ, which is important for the finiteness of the energy.

Now we cannot neglect the terms which are small in a formal perturbation expansion as we are looking for solutions which cannot be obtained in perturbation theory. We also note that in practice this construction is applied to quantum chromodynamics, where the coupling constant g is not small.

We shall use the t'Hooft-Polyakov ansatz

$$A_j^i(x) = \varepsilon^{ijk}\frac{x^k}{r}W(r); \quad \varphi^i(x) = \delta^{ji}\frac{x_j}{r}F(r)$$
$$\chi^i(x) = \delta^{ji}\frac{x_j}{r}G(r); \quad A_0^i(x) = 0,$$
$$r \to \infty, W(r) \to (gr)^{-1}, F(r) \to F\cosh\gamma, G(r) \to F\sinh\gamma,$$
$$F\cosh\gamma = \frac{m}{g}; \quad F\sinh\gamma = -\alpha\frac{m}{g}. \qquad (16)$$

We may consider the theories, where g is not small (as QCD). Now the parameter α is different from 1. In these cases the parameters F, γ are finite.

The equations (15) may be rewritten in terms of the functions

$$K(r) = 1 - grW(r); \quad J(r) = F(r)rg; \quad Y(r) = G(r)rg \tag{17}$$

$$r^2 \frac{d^2K}{dr^2} = (K^2 + J^2 - Y^2 - 1)K(r); \quad K(r) \to 0, r \to \infty$$
$$r^2 \frac{d^2J}{dr^2} = 2K^2 J; \quad J(r) \to Frg \cosh\gamma; r \to \infty$$
$$r^2 \frac{d^2Y}{dr^2} = 2K^2 Y; \quad Y(r) \to Frg \sinh\gamma = -\alpha Frg \cosh\gamma; r \to \infty \tag{18}$$

Following the paper be B.Julia and A.Zee we take the following ansatz for the solutions

$$J(r) = \Lambda(r)\cosh\gamma; \quad Y(r) = \Lambda(r)\sinh\gamma;$$
$$\Lambda(r)\cosh\gamma \to Frg\cosh\gamma; \quad \Lambda(r)\sinh\gamma \to Frg\sinh\gamma. \qquad (19)$$

Thus the equations (18)acquire the form

$$r^2\frac{d^2K}{dr^2} = (K^2 + \Lambda^2 - 1)K; \quad K \to 0, r \to \infty,$$
$$r^2\frac{d^2\Lambda}{dr^2} = 2K^2\Lambda; \quad \Lambda(r) \to Frg; \quad r \to \infty. \qquad (20)$$

The solutions of these equations are well known

$$K(r) = \frac{rgF}{\sinh rgF}; \quad \Lambda(r) = \frac{rgF}{\tanh grF} - 1. \tag{21}$$

Obviously these solutions possess positive and limited energy, namely they have the same energy as the magnetic monopole

$$E = \int d^3x[\frac{1}{4}F^i_{lm}F^i_{lm} + \frac{1}{2}(D_l\varphi)^i(D_l\varphi)^i - \frac{1}{2}(D_l\chi)^i(D_l\chi)^i] =$$
$$\int d^3x[\frac{1}{4}F^i_{lm}F^i_{lm} + \frac{1}{2}(D_l\Lambda)^i(D_l\Lambda^i)] \tag{22}$$

Using the gauge invariant definition for electromagnetic field tensor

$$F_{\mu\nu} = \widehat{\Lambda}^i F^i_{\mu\nu} - g^{-1}\varepsilon^{ijk}\widehat{\Lambda}^i (D_\mu \widehat{\Lambda})^j (D_\nu \widehat{\Lambda})^k \tag{23}$$

where $\widehat{\Lambda}^i = \frac{\Lambda^i}{|\Lambda|}$; $|\Lambda^| = (\sum_i \Lambda^i \Lambda^i)^{1/2}$ we found the excitation we consider is the magnetic monopole, which produces the magnetic field

$$B^i(x) = \frac{x^i}{gr^3} \tag{24}$$

One sees that even for g large the mass and magnetic field of monopole solution do not depend on γ, and are determined by the constants F and g. The solution (19-21)has no electric charge. It is easy to include into this scheme also the excitations possessing electric and magnetic charges, dyons.

Conclusion

So we showed that the modified formulation of the Yang-Mills theory indeed admits soliton excitations. Let us remind here our starting points. We are looking for the nonperturbative soliton solutions which corresponds to the gauge invariant Lagrangian generating the same perturbation series as the standard Yang-Mills theory. The on-shell matrix elements of the scattering matrix in perturbation theory formally coincide, but for QCD perturbatively do not exist due to infrared divergencies. The correlation functions of the gauge invariant operators are free of this problem and coincide in the framework of perturbation theory in the both theories.

Modified formulation in the topologically nontrivial sector has the classical solutions, corresponding to solitons. Of course our treatment of solitons was purely classical and there are many questions to be answered. The main question is related to the existence in our formulation of negative energies. In perturbation theory we were able to show that the negative energy states decouple from the positive energy ones. Beyond perturbation theory it is an open question. There is no problem to introduce the indefinite metrics, which makes the eigenvalues of the Hamiltonian positive, but the question about the existence of transitions between different states is open.