Non-equilibrium dynamics of integrable models: Brief review and current research projects

Balázs Pozsgay

Eötvös Loránd University Budapest
MTA-ELTE "Momentum" Integrable Quantum Dynamics Research group

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The topic of this talk

- History of integrable models

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- The last decade: Non-equilibrium dynamics

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- History of integrable models
- The last decade: Non-equilibrium dynamics
- Ongoing research projects and future plans

Integrability

What is it??

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"Integrable models are special many body systems whose exact solution is possible with analytic methods".

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- Scattering: Elastic and factorized (two-body reducible)
- Integrable models: between free and generic interacting models

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- Three body scattering:
  product of two body scattering events

Further history

- Bethe Ansatz for the 1D Bose gas:

$$H = \sum_{j=1}^{N} -\frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<k} \delta(x_j - x_k)$$

Lieb and Liniger, 1963

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- Integrable S-matrices in 1+1 QFT: sine-Gordon model, etc...
  (A. B. Zamolodchikov & Al. B. Zamolodchikov, ...)

Quantum Inverse Scattering Method
(L. Faddeev and the Leningrad group, around 1980)

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- Standard and unified framework
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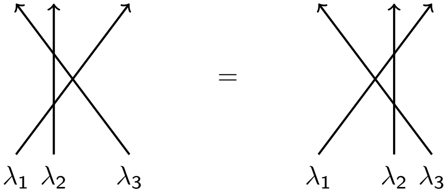
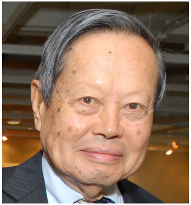
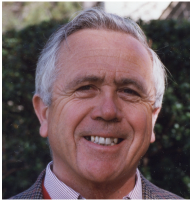
- Yang-Baxter relation:







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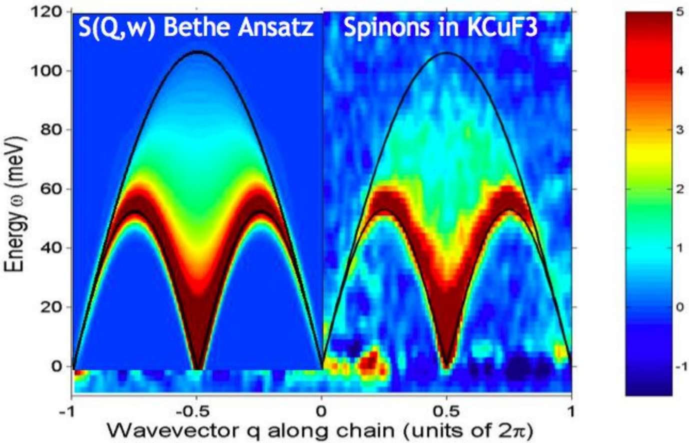
- Relation to AdS/CFT

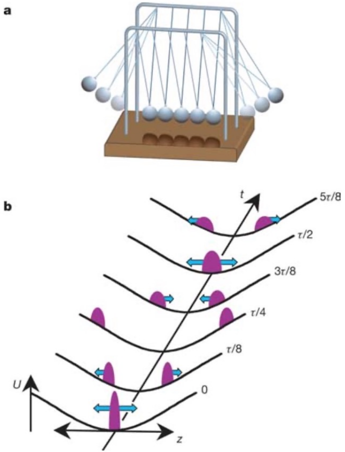
Experiments

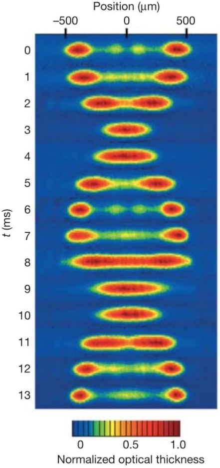


B. Lake, D. A. Tennant, J.-S. Caux, T. Barthel, U. Schollwöck, S. E. Nagler, and C. D. Frost, *Multispinon Continua at Zero and Finite Temperature in a Near-Ideal Heisenberg Chain*, Phys. Rev. Lett. 111, 137205

Quantum simulation of integrable models!

Quantum Newton's Cradle





T. Kinoshita, T. Wenger and D. S. Weiss, *A quantum Newton's cradle*, Nature 440, 900–903 (2006)

Non-equilibrium dynamics

- Equilibration and thermalization (?)

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- Transport properties

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- Generalized Gibbs Ensemble (GGE, M. Rigol, 2006):

$$\rho_{GGE} = \frac{e^{-\sum_j \beta_j Q_j}}{Z}$$

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$$\langle\Psi_0|Q_\alpha|\Psi_0\rangle = \mathrm{Tr}\left(\rho_{GGE} Q_\alpha\right)$$

- How to give predictions for the dynamics?

$$\langle\Psi_0|e^{iHt}\mathcal{O}e^{-iHt}|\Psi_0\rangle = ?$$

GGE for the Heisenberg spin chains

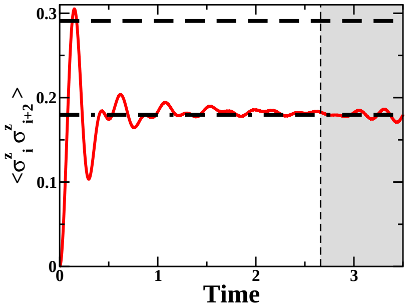
- Local charges Q_α were known from QISM

GGE for the Heisenberg spin chains

- Local charges Q_α were known from QISM
- Local charges not complete for the GGE!

  B. Wouters, J. De Nardis, M. Brockmann, D. Fioretto, M. Rigol, and J.-S. Caux Phys. Rev. Lett. 113, 117202 (2014)

  B. Pozsgay, M. Mestyán, M. A. Werner, M. Kormos, G. Zaránd, and G. Takács, Phys. Rev. Lett. 113, 117203 (2014)



GGE in the Heisenberg spin chains

- Solution: quasi-local charges

$$Q = \sum_{x} q(x), \qquad q(x) = \sum_{r} q^{(r)}(x)$$

E. Ilievski, M. Medenjak, T. Prosen, Phys. Rev. Lett. 115, 120601 (2015)

E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, T. Prosen Phys. Rev. Lett. 115, 157201 (2015)

Generalized Hydrodynamics (GHD)

- Transport in the large time, long distance limit

  O. A. Castro-Alvaredo, B. Doyon, T. Yoshimura, Physical Review X 6 (4), 041065, (2016)

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- Describes ballistic, diffusive, even super-diffusive transport
- Fluid cells: Important to know $\langle q_\alpha(x,t)\rangle$

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Generalized Hydrodynamics (GHD)

- New finite volume result:

$$\langle \boldsymbol{\lambda}_N | J_\alpha(x) | \boldsymbol{\lambda}_N \rangle = \frac{1}{L} \sum_j h_\alpha(\lambda_j) v_{\text{eff}}(\lambda_j), \qquad \mathbf{v}_{\text{eff}} = L G^{-1} \mathbf{e}'$$

where

$$G_{ij} = \delta_{ij} L p'(\lambda_j) + \Big(\dots \delta'() \dots \Big)$$

M. Borsi, BP, L. Pristyák, Phys. Rev. X 10, 011054 (2020)
BP, SciPost Phys. 8, 016 (2020)
BP, Phys. Rev. Lett. 125, 070602 (2020)

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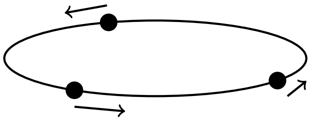
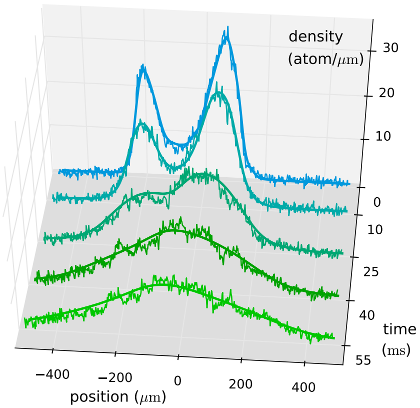
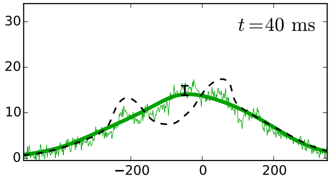
- In free models: $v = \frac{e'}{p'} = \frac{de}{dp}$. Interaction modifies it: $\sim \delta'(p_j, p_k)$



Generalized HydroDynamics on an Atom Chip, Max Schemmer et.al, Phys. Rev. Lett. 122, 090601 (2019)





Ongoing and future projects

Currents and GHD in the XYZ model

$$H = \sum_j J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z$$

Entanglement production?

Ongoing and future projects

Overlap computations

$$\langle \Psi_0 | \boldsymbol{\lambda}_N \rangle = ?$$

Relevant for AdS/CFT

M. de Leeuw, T. Gombor, C. Kristjansen, G. Linardopoulos, BP,
JHEP 2020, 176

Integrability breaking

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- Lindblad equation:

$$\dot{\rho} = i[\rho, H] + \sum_a \gamma_a \left[L_a \rho L_a^\dagger - \frac{1}{2}\{L_a^\dagger L_a, \rho\} \right]$$

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- Question:

$$\frac{d}{dt} \langle q_\alpha(x) \rangle =?$$

Thanks for the attention!

Generalized Hydrodynamics (GHD)

- Fluid cells: $\rho(\lambda|x, t)$

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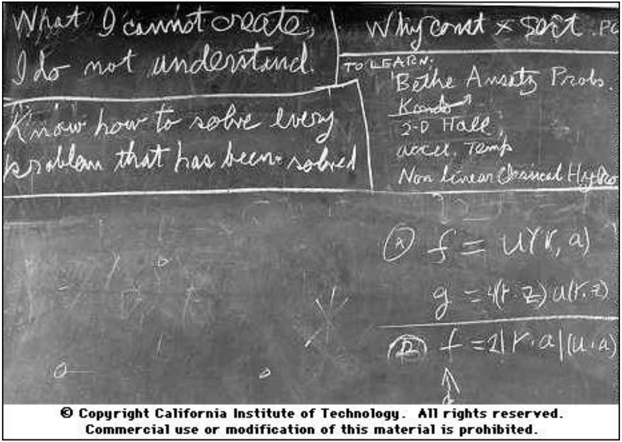
- The ballistic flow equation becomes:

$$\partial_t \rho(\lambda|x,t) + \partial_x (v_{\text{eff}}(\lambda|x,t)\rho(\lambda|x,t)) = 0$$

"I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better."

Richard P. Feynman, 1988



GGE

- Bethe Ansatz: In finite volume $\boldsymbol{\lambda}_N$

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- Completeness, if

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- $\rho(\lambda)$ not local, not quasi-local.
  However, the GGE can be quasi-local.