Electrodynamics at the classical electron radius and the Abraham-Lorentz force[1]

1. Subclassical island in the quantum domain:
   Abraham-Lorentz force as a saddle point effect of QED
2. Divergence and instability of relativistic, non-local field theories
3. Classical effective theory and loop graphs
4. Numerical indication of the stability of electrodynamics

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Scale hierarchy in QED

Dimensionless $\alpha = \dfrac{e^2}{\hbar c} \Longrightarrow$ scale hirarchy: $r_n = \alpha^n r_0$

Bohr radius: $r_{-2} = a_0 = \frac{\hbar^2}{me^2}$ absence of c: non-relativistic QM

Compton wavelength: $r_{-1} = \frac{\hbar}{mc}$ absence of e: pair creation

Classical electron radius: $r_0 = \frac{e^2}{mc^2}$ absence of $\hbar$: subclassical physics

no decoherence

$r_0 = 2.8 fm$

strong field of point charges

self interaction $\Longrightarrow$ AL force

Lamb shift: $r_1 = \alpha^3 a_0 = \frac{e^4}{\hbar mc^2}$ vacuum polarization

$n \geq 2$: covered Standard Model

$n \leq -3$: covered by many-body effects (collective modes & chemistry)

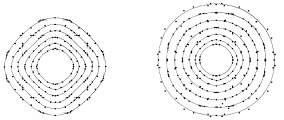
Regulators in QFT

Requirements

Non-perturbative:

- applied to the path integral rather than to the propagator
- dimensional regularization

Symmetries:

- local symmetries must be preserved
- global symmetries migh be broken at the cutoff scale
- Lorentz symmetry:  (C. Lang, C. Rebbi 1982)
- strong field of point particles needs intact Lorentz symmetry
- lattice, non-rel. momentum cutoff

Definite norm:

- higher order time derivatives, Pauli-Villars, proper time methods

Stability:

- higher derivative theories are unstable M. Ostrogradsky 1850
- higher order time derivatives, Pauli-Villars, proper time methods

Causality:

- higher order time derivatives, Pauli-Villars, proper time methods

Divergences of Lorentz symmetry

Lorentz invariant integral:

$$\int d^d p f(p^2) = \int d^d p f(p^2) \underbrace{\int ds^2 \delta(p^2 - s^2)}_{1} = \int ds^2 f(s^2) V(s^2)$$

$$V(s^2) = \int_{-\Lambda}^{\Lambda} d^d p \delta(p^2 - s^2) \sim \begin{cases} \Lambda^{d-2} & d > 2 \\ \ln \Lambda & d = 2 \end{cases}$$

Non-compact external symmetry group:
- momentum space: UV divergence
- space-time: IR divergence

F. Dyson 1949: conditional convergence
W. Zimmermann 1968: $i\epsilon \to i\epsilon(\boldsymbol{p}^2 + m^2)$

$\Longrightarrow$ No relativistic free field theory with finite Green's functions

Point-splitting regularization is still possible

UV cutoff $\leftrightarrow$ non-locality

Local dynamics:
Constants of motion: $\mathcal{C} = \{C_j(x, \dot{x}, t)\}$
Conserved quantities (Noether's theorem) $\mathcal{Q} \subset \mathcal{C}$
Integrals of the motion $\mathcal{I} = \{I_j(x, \dot{x})\} \subset \mathcal{C}$
Equation of motion: $\mathcal{Q} \subset \mathcal{I}$
Ballance equations: $\partial_t F(x, \dot{x}) = \rho(x, \dot{x}), \quad \{\partial_t F - \rho\} \subset \mathcal{I} \backslash \mathcal{Q}$

Non-local dynamics:

$$S = \sum_{n \geq 1} \frac{1}{n!} \int_V dx_1 \cdots dx_n L^{(n)}(x_1, \phi(x_1), \partial\phi(x_1), \ldots, x_n, \phi(x_n), \partial\phi(x_n))$$

- Dependence on x_j is important $\Longrightarrow$ no integral of motion
- Conservation law without symmetry:
  particle dynamics: $\partial_t F = \rho \Longrightarrow F' = F - \int_{t_i}^{t} dt' \rho$
  field theory: $\partial_\mu j^\mu = \rho \Longrightarrow j'^\mu(x) = j^\mu(x) + \int dy D_0^r(x-y)\partial^\mu \rho(y)$
  $(\Box D_0 = -1)$

Non-local theories

Stability

Action:

$$S = \sum_{n\geq 1} \frac{1}{n!} \int_V dx_1 \cdots dx_n L^{(n)}(x_1, \phi(x_1), \partial\phi(x_1), \ldots, x_n, \phi(x_n), \partial\phi(x_n))$$

Energy-momentum tensor:

$$T^{\mu\nu} = \sum_n \frac{1}{(n-1)!} \int_V dx_2 \cdots dx_n \left(\frac{\partial L^{(n)}}{\partial \partial_\mu \phi_1} \partial_\nu \phi_1 - L^{(n)} g^\mu_\nu \right),$$

Ballance equation: $\partial_\mu T^{\mu\nu} = \rho^\nu$

$$\rho_\mu = -\sum_n \frac{1}{(n-1)!} \int_V dx_2 \cdots dx_n \partial_{x_1^\mu} L^{(n)}.$$

Conserved EM tensor:

$$T'^{\mu\nu}(x) = T^{\mu\nu}(x) + \int dy D_0^r(x-y) \partial^\mu \rho^\nu(y).$$

Energy is either non-conserved (Newton III. is violated) or non-definite

Are there interacting relativistic field theories?

AL force as saddle point dynamics

Effective current dynamics

Elimination of dynamical degrees of freedom $\Longrightarrow$ CTP (CQCO formalism)

Unreadable final state (radiation):

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{a}]} &= \mathrm{Tr}[U[a^+]\rho_i U^\dagger[-a^-]] \\
&= \int D[\hat{\bar{\psi}}]D[\hat{\psi}]D[\hat{A}]e^{\frac{i}{\hbar c}\int dx \hat{\bar{\psi}} G^{-1}\hat{\psi} + \frac{i}{2c\hbar}\int dx \hat{A}\hat{D}_A^{-1}\hat{A} + \frac{i}{\hbar c}\int dx \hat{J}(e\hat{\sigma}\hat{A}+\hat{a})}
\end{aligned}
$$

Photon propagator: $\hat{D}_A^{\mu\nu} = -4\pi\hat{D}(m=0)T^{\mu\nu}$ (Landau gauge)

$$
\hat{D}_k(m) = \begin{pmatrix} \frac{1}{k^2 - \frac{m^2c^2}{\hbar^2} + i\epsilon} & -2\pi i\delta(k^2 - \frac{m^2c^2}{\hbar^2})\Theta(-k^0) \\ -2\pi i\delta(k^2 - \frac{m^2c^2}{\hbar^2})\Theta(k^0) & -\frac{1}{k^2 - \frac{m^2c^2}{\hbar^2} - i\epsilon} \end{pmatrix}
$$

Electron propagator: $\hat{G}_k = (\not{k} + \frac{mc}{\hbar})\hat{D}_k$

Electric current: $J^{\mu\sigma} = \bar{\psi}^\sigma\gamma^\mu\psi^\sigma$

Simplectic "metric tensor": $\hat{\sigma} = \mathrm{Diag}(1, -1)$

AL force as saddle point dynamics

Point splitting regularization

Scalar theory:

$$
\begin{aligned}
S &= \sum_a \int dx \left\{ \frac{1}{2}\{a[\partial\phi^a(x)]^2 - (am^2 - i\epsilon)\phi^{a2}(x)\} - aU(\phi^a(x))\} \right\} \\
\hat\phi_\Lambda(x) &= \hat\chi\hat\phi(x) = \int dy \hat\chi(x-y)\hat\phi(y), \quad \hat\chi = [-\Lambda^2\hat D_\Lambda]^{n_\Lambda} \\
Z_B[\hat j] &= \int D[\hat\phi] e^{\frac{i}{\hbar}[\frac{1}{2}\hat\phi\hat D^{-1}\hat\phi - \int dx [U_B(\phi^+_\Lambda) - U_B(\phi^-_\Lambda)] + \int dx \hat j \hat\phi]}
\end{aligned}
$$

Equivalent form: $\hat\phi \to \hat\phi_B$, $\hat D_B = \hat\chi\hat D\hat\chi$

$$
Z_B[\hat j] = \int D[\hat\phi_B] e^{\frac{i}{\hbar}[\frac{1}{2}\hat\phi_B\hat D_B^{-1}\hat\phi_B - \int dx [U_B(\phi^+_B) - U_B(\phi^-_B)] + \int dx \hat j \chi^{-1}\hat\phi_B]}
$$

- $D^{\overset{r}{a}}_\Lambda = \chi^{\overset{r}{a}} D^{\overset{r}{a}} \chi^{\overset{r}{a}}$: causality
- Point splitting = higher order derivative except at the external legs

QED: $S_B = S_M[\hat A] + S_D[\hat{\bar\psi}, \hat\psi] + S_i[\hat{\bar\psi}_B, \hat\psi_B, \hat A_B]$,
$\hat A_B = \hat\kappa\hat A$, $\hat\psi_B = \hat\chi[\hat A]\hat\psi$, $\hat{\bar\psi}_B = \hat{\bar\psi}\hat{\bar\chi}^{-1}[\hat A]$
Gauge invariance: $\partial_\mu \to D_\mu = \partial_\mu + ieLA_\mu$ in $\hat\chi$.

AL force as saddle point dynamics

Tree-level effective current dynamics

Connected Green's functions:

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{a}]} &= \int D[\hat{\bar{\psi}}]D[\hat{\psi}]D[\hat{A}]e^{\frac{i}{\hbar}\hat{\bar{\psi}}\hat{G}^{-1}\hat{\psi}+\frac{i}{2\hbar}\hat{A}\hat{D}_A^{-1}\hat{A}+\frac{i}{\hbar}\hat{J}(e\hat{\sigma}\hat{A}+\hat{a})+\frac{i}{\hbar}\hat{\bar{\eta}}\hat{\psi}+\frac{i}{\hbar}\hat{\bar{\psi}}\hat{\eta}} \\
&= \int D[\hat{\bar{\psi}}]D[\hat{\psi}]e^{\frac{i}{\hbar}\hat{\bar{\psi}}[\hat{G}^{-1}+\hat{\not{a}}]\hat{\psi}-\frac{i}{2\hbar}\hat{J}\hat{D}_A\hat{J}+\frac{i}{\hbar}\hat{\bar{\eta}}\hat{\psi}+\frac{i}{\hbar}\hat{\bar{\psi}}\hat{\eta}} \\
&= e^{\frac{i\hbar e^2}{2}\frac{\delta}{\delta\hat{a}}\hat{D}_A\frac{\delta}{\delta\hat{a}}}e^{\frac{i}{\hbar}W_0[\hat{a}]} \\
e^{\frac{i}{\hbar}W_0[\hat{a}]} &= \int D[\hat{\bar{\psi}}]D[\hat{\psi}]e^{\frac{i}{\hbar c}\int dx\hat{\bar{\psi}}[\hat{G}^{-1}+\hat{\not{a}}]\hat{\psi}+\frac{i}{\hbar}\hat{\bar{\eta}}\hat{\psi}+\frac{i}{\hbar}\hat{\bar{\psi}}\hat{\eta}} \\
W_0[\hat{a}] &= -\hat{\bar{\eta}}\frac{1}{\hat{G}^{-1}-\hat{\not{a}}}\eta - i\hbar\mathrm{Tr}\ln[\hat{G}^{-1}-\hat{\not{a}}]
\end{aligned}
$$

Effective action: $\Gamma[\hat{j}] = W[\hat{a}] - \hat{a}\hat{j}, \quad \hat{j} = \frac{\delta W[\hat{a}]}{\delta\hat{a}}$ **E.o.M:** $\frac{\delta\Gamma[\hat{j}]}{\delta\hat{j}} = -\hat{a}$

Non-local non-polynomial action without small parameter!

Assume a free world-line dynamics:

$$
\begin{aligned}
\hat{j}(x) &= \frac{\delta W_0[\hat{a}]}{\delta\hat{a}(x)} = c\int ds\delta(x-x^a(s))\dot{x}^{a,\mu}(s) \qquad (\dot{x}^0>0) \\
\Gamma_0[\hat{j}] &= W_0[\hat{a}] - \hat{a}\hat{j} = -m_B c\int ds(\sqrt{\dot{x}^{+2}} - \sqrt{\dot{x}^{-2}})
\end{aligned}
$$

AL force as saddle point dynamics

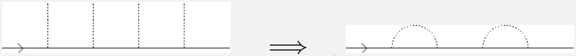
Tree-level effective current dynamics

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{a}]} &= e^{\frac{i\hbar e^2}{2c}\int dxdy \frac{\delta}{\delta \hat{a}(x)}\hat{D}_A(x-y)\frac{\delta}{\delta \hat{a}(y)}} e^{\frac{i}{\hbar}W_0[\hat{a}]} \\
W[\hat{a}] &= W_0[\hat{a}] - \frac{e^2}{2c}\int dxdy \hat{j}(x)\hat{D}_A(x-y)\hat{j}(y) + \mathcal{O}(\hbar) \\
\Gamma[\hat{j}] &= \Gamma_0[\hat{j}] - \frac{e^2}{2c}\int dxdy \hat{j}(x)\hat{D}_A(x-y)\hat{j}(y) + \mathcal{O}(\hbar)
\end{aligned}
$$

Tree-level saddle point dynamics: action-at-a-distance theory (Schwarzschild-Tetrode-Fokker-Wheeler-Feynman) with retardation

Classical effective theories:

1. E.o.M.: Tree $\Longrightarrow$ loop graphs



N loops, N classical lines $\Longrightarrow \frac{\hbar^N}{\hbar^N}$

2. Non-local radiation field $\Longrightarrow$ non-local action

AL force as saddle point dynamics

Classical effective CTP action

System: ϕ, environment: ψ

$$\begin{aligned} S[\phi,\psi] &= S_s[\phi] + S_e[\phi,\psi] \\ S[\hat\phi,\hat\psi] &= S_s[\phi^+] + S_e[\phi^+,\psi^+] - S_s[\phi^-] - S_e[\phi^-,\psi^-] \end{aligned}$$

Effective action: $\frac{\delta S[\hat\phi,\hat\psi]}{\delta\hat\psi} = 0 \Longrightarrow \hat\psi = \hat\psi[\hat\phi]$

$$\begin{aligned} S_{eff}[\hat\phi] &= S[\hat\phi,\hat\psi[\hat\phi]] \\ &= S_s[\phi^+] + S_e[\phi^+,\psi^+[\hat\phi]] - S_s[\phi^-] - S_e[\phi^-,\psi^-[\hat\phi]] \\ &= S_s[\phi^+] - S_s[\phi^-] + S_{infl}[\hat\phi] \end{aligned}$$

Influence functional *(Feynman, Vernon 1963)*:

$$S_{infl}[\hat\phi] = S_e[\phi^+, y^+[\hat\phi]] - S_e[v^-, y^-[\hat\phi]]$$

Better parametrization:

$$S_{eff}[\hat\phi] = S_1[\phi^+] - S_1[\phi^-] + S_2[\hat\phi], \quad (S_2[0,\phi^-] = S_2[\phi^+,0] = 0)$$

AL force as saddle point dynamics

Classical effective CTP action

Keldysh parametrization: $\phi^{\pm} = \phi \pm \frac{\phi^d}{2}$ *(Keldysh 1964)*
Advantage: $\phi^+ = \phi^-$, $\phi^d = 0 \Longrightarrow S = \mathcal{O}\left(\phi^d\right)$ is sufficient

E.o.M for x^d:

$$\begin{aligned} 0 &= \frac{\delta}{\delta\phi^d}\left\{S_1\left[\phi + \frac{\phi^d}{2}\right] - S_1\left[\phi - \frac{\phi^d}{2}\right] + S_2\left[\phi + \frac{\phi^d}{2}, \phi - \frac{\phi^d}{2}\right]\right\}_{|\phi^d=0} \\ &= \frac{\delta S_1[\phi]}{\delta\phi} + \frac{\delta S_2[\phi^+, \phi^-]}{\delta\phi^+}_{|\phi^+=\phi^-=\phi} \end{aligned}$$

↑
semiholonomic forces

- S_1: Holonomic forces (Noether theorem available)
- S_2: Semiholonomic forces, environment excitations
- $L = \frac{m}{2}\dot{x}^{+2} - \frac{m\omega^2}{2}x^{+2} - \frac{m}{2}\dot{x}^{-2} + \frac{m\omega^2}{2}x^{-2} + \frac{k}{2}(x^-\dot{x}^+ - x^+\dot{x}^-)$
  EoM: $m\ddot{x}^{\pm} = -m\omega^2 x^{\pm} - k\dot{x}^{\mp}$ *(Bateman 1931)*

AL force as saddle point dynamics

Tree-level effective current dynamics

Action:

$$\Gamma_{tree} = S_m + S_{infl}$$

$$S_m[\hat{x}] = -m_B c \int ds(\sqrt{\dot{x}^{+2}(s)} - \sqrt{\dot{x}^{-2}(s)}) + \int ds[\dot{x}^+(s) - \dot{x}^-(s)]k(s)$$

$$S_{infl}[\hat{x}] = -\frac{e^2}{2c}\sum_{ab} ab \int ds ds' \dot{x}^{a,\mu}(s) D_A^{ab}(x^a(s) - x^b(s'))\dot{x}^b_{\mu'}(s')$$

$\mathcal{O}\left(x^d\right)$:

$$\mathrm{Re} S_{infl}[\hat{x}] = -\frac{e^2}{c}\int ds ds' \{\dot{x}'\dot{x}^d D_A^r(x(s) - x(s')) + (x^d\partial)D_A^r(x(s) - x(s'))\}$$

Regularization: $D^r(x) = -\Theta(x^0)\frac{\delta(x^2)}{2\pi} \to -\Theta(x^0)\frac{\delta_\Lambda(x^2)}{2\pi}$

Conditions: $\int_0^\infty dz \delta_\Lambda(z) = 1$ and $\delta_\Lambda(0) = 0$

Examples: $\delta_\Lambda(x^2) = \delta(x^2 - \ell^2)$ or $\delta_\Lambda(x^2) = \frac{\Theta(x^2)}{12\ell^4}x^2 e^{-\frac{\sqrt{x^2}}{\ell}}$

AL force as saddle point dynamics

Linearized equation of motion

Non-local action with memory:

$$\mathrm{Re}S^{(2)}_{infl} = \frac{2e^2}{c}\int_{-\infty}^{\infty} ds x^d \int_{-\infty}^{0} du\delta_\Lambda(u^2)\frac{u^2\ddot{x}' + x' - x - u\dot{x}'}{u^2}$$

$\delta_\Lambda(u^2)$: Non-uniform convergence as $\frac{1}{\ell} = \Lambda \to \infty$: $L_{sing} \neq 0$, $L_{reg} \to 0$

$$L_{reg} = \frac{2e^2}{c}x^d\int_{-\infty}^{0} du\delta_\Lambda(u^2)\frac{x' - x - u\dot{x}' + u^2\left(\ddot{x}' - \frac{1}{2}\ddot{x}\right) - \frac{2u^3}{3}\dddot{x}}{u^2}$$

$$L_{sing} = \frac{e^2}{c}x^d\int_{-\infty}^{0} du\delta_\Lambda(u^2)\left(\ddot{x} + \frac{4u}{3}\dddot{x}\right)$$

↗ ↑

mass renormalization | AL force: "anomaly" effect

↘

External source:

$$\dot{x}^\mu(s) = -\int_{-\infty}^{\infty} ds' F(s-s')k^\mu(s')$$

$$F(\omega) = \frac{1}{(\omega+i\epsilon)^2\chi(\omega)}, \qquad \chi(\omega) = 1 + r_0\omega\left[\frac{2}{3}i + \mathcal{O}\left(\frac{\omega}{\Lambda}\right)\right].$$

AL force as saddle point dynamics

Full equation of motion

E.o.M:

$$\ddot{x} = r_0 \frac{m}{m_B} \int_{-\infty}^{0} du \delta'_\Lambda((x-x')^2)\{(x-x')(\dot{x}\dot{x}') - [\dot{x}(x-x')]\dot{x}']\}$$

Regularization:

- $\delta_\Lambda(x^2) = \delta(x^2 - \ell^2)$:

$$\begin{aligned} \ddot{x} \quad &= \quad \frac{m}{m_B} \frac{r_0}{[\dot{x}'(x-x')]^2} \left[\frac{\ddot{x}'(x-x') - 1}{\dot{x}'(x-x')} \{(x-x')(\dot{x}\dot{x}') - [\dot{x}(x-x')]\dot{x}'\} \right. \\ & \left. -(x-x')(\dot{x}\ddot{x}') + [\dot{x}(x-x')]\ddot{x}' \right] \end{aligned}$$

- $\delta_\Lambda(x^2) = \frac{\Theta(x^2)}{12\ell^4} x^2 e^{-\frac{\sqrt{x^2}}{\ell}}$:

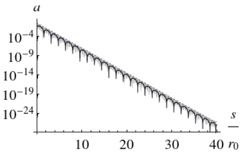
$$\begin{aligned} \ddot{x} \quad &= \quad \frac{r_0}{3\ell^4} \frac{m}{m_B} \int_{-\infty}^{0} du \left(1 - \frac{\sqrt{(x-x')^2}}{2\ell}\right) e^{-\frac{\sqrt{(x-x')^2}}{\ell}} \\ & \times \{(x-x')(\dot{x}\dot{x}') - [\dot{x}(x-x')]\dot{x}']\}. \end{aligned}$$

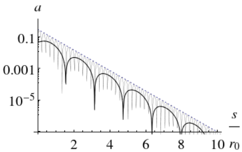
AL force as saddle point dynamics

Full equation of motion

Renormalization conditions: e, m, AL pole $F(\omega) = \frac{1}{(\omega+i\epsilon)^2(1+\frac{2}{3}ir_0\omega)}$
Free bare parameters: m_B, ℓ

Scaling of $a = |\ddot{x}|r_0$ for the two renormalized trajectories:





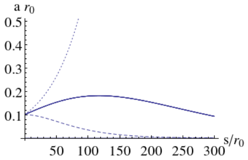
$(- : \ \ell = \frac{r_0}{3}$ and $\cdots : \ \ell = \frac{r_0}{15})$

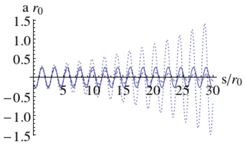
Smooth cutoff $\Longrightarrow$ Zitterbewegung

AL force as saddle point dynamics

Full equation of motion

Accelereation: $a = |\ddot{x}| r_0$

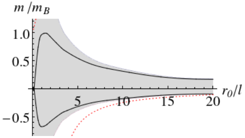


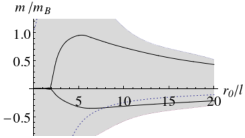


$$\delta_\Lambda(x^2) = \delta(x^2 - \ell^2)$$

$$\delta_\Lambda(x^2) = \frac{\Theta(x^2)}{12\ell^4} x^2 e^{-\frac{\sqrt{x^2}}{\ell}}$$

Domain of stability and renormalized trajectories by keeping the AL pole:





Summary

1. Subclassical island in the quantum domain of QED
   Abraham-Lorentz force as a saddle point effect of QED
2. Relativistic kinematics generates divergent loop integrals
   Are there relativistic field theories?
3. Effective theories:
   - Mass-shell radiation modes: non-locality
   - Elimination of degrees of freedom: loop graphs
4. QED:
   - Domain of stability
   - Renormalized trajectories
   - No Landau pole on the tree-level
5. Instability at other scales?
   - Needs unconstrained final state $\Longrightarrow$ CTP
   - Unbounded runaway modes $\Longrightarrow$ no Fourier integrals in frequency space
   - Residue theorem $\Longrightarrow$ acausal poles in D^r, not seen in D_F $\Longrightarrow$ CTP