The $f_{PS,V}/m_V$ ratios and the conformal window

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Context and motivation

- Non-abelian gauge theories (G, N_f, R) in 4 dimensions, $m = 0$

- N_f small: $0 \leq N_f < N_f^*$

  chiral symmetry spontaneously broken, Λ

- N_f large but not very large

  $N_f^* < N_f < N_f^{asympt}$: CFT $\rightarrow$ conformal window

- Main question: what is N_f^* for given G and R?

  lower end of the conformal window

- Our work: $G = SU(3)$ and $R = fund$

Motivation

- N_f^* potentially determined by non-perturbative physics

- N_f^{asympt} purely perturbative (1-loop)

- $G = SU(3), \quad R = fund: \quad N_f^{asympt} = 16.5$

  from 1-loop β-function

Motivation

- No clear consensus on N_f^* for $G = SU(3)$ and $R = fund$
- Somewhere around $8 - 13$
- Lattice would be ideal, but very costly: large finite volume effects, large systematic errors, need for large statistics, ...
- All kinds of not "ab initio" approaches instead
- Our approach will also be speculative somewhat, but combine both **perturbative** and **non-perturbative** physics

Motivation

- **Perturbative** calculations: reliable close to $N_f^{asympt} = 16.5$
- **Non-perturbative** calculations: for low $0 < N_f < 11$
- Combine both in a meaningful way
- $f_{PS,V}/m_V$ ratio across full range $0 < N_f \leq 16.5$
- Observe abrupt change at some $N_f \rightarrow$ identify with N_f^*

Setup

Define $f_{PS,V}$ and m_V at finite fermion mass m

Regardless of N_f they are finite and scheme independent (physical)

Well-defined ratios for all $N_f \leq 16.5$

Setup

Chiral limit - below conformal window

$$f_{PS}, \;\; f_V, \;\; m_V \sim \Lambda$$

Ratio $f_{PS,V}/m_V = O(\Lambda)/O(\Lambda) = const$ finite

Setup

Chiral limit - inside conformal window

$$f_{PS}, \;\; f_V, \;\; m_V \sim m^{\alpha}$$

With the *same* $\alpha = \frac{1}{1+\gamma}$

Ratio $f_{PS,V}/m_V = O(m^{\alpha})/O(m^{\alpha}) = const$ finite

Setup

The ratios are well-defined in the chiral limit for all $N_f \leq 16.5$

$f_{PS,V}/m_V$ in the chiral limit is just a function of N_f

Low N_f, from past lattice work

- f_{PS}/m_V in chiral, continuum limit for $2 \leq N_f \leq 10$
- Largely N_f-independent
- Some constant
- f_V from f_{PS} using KSRF

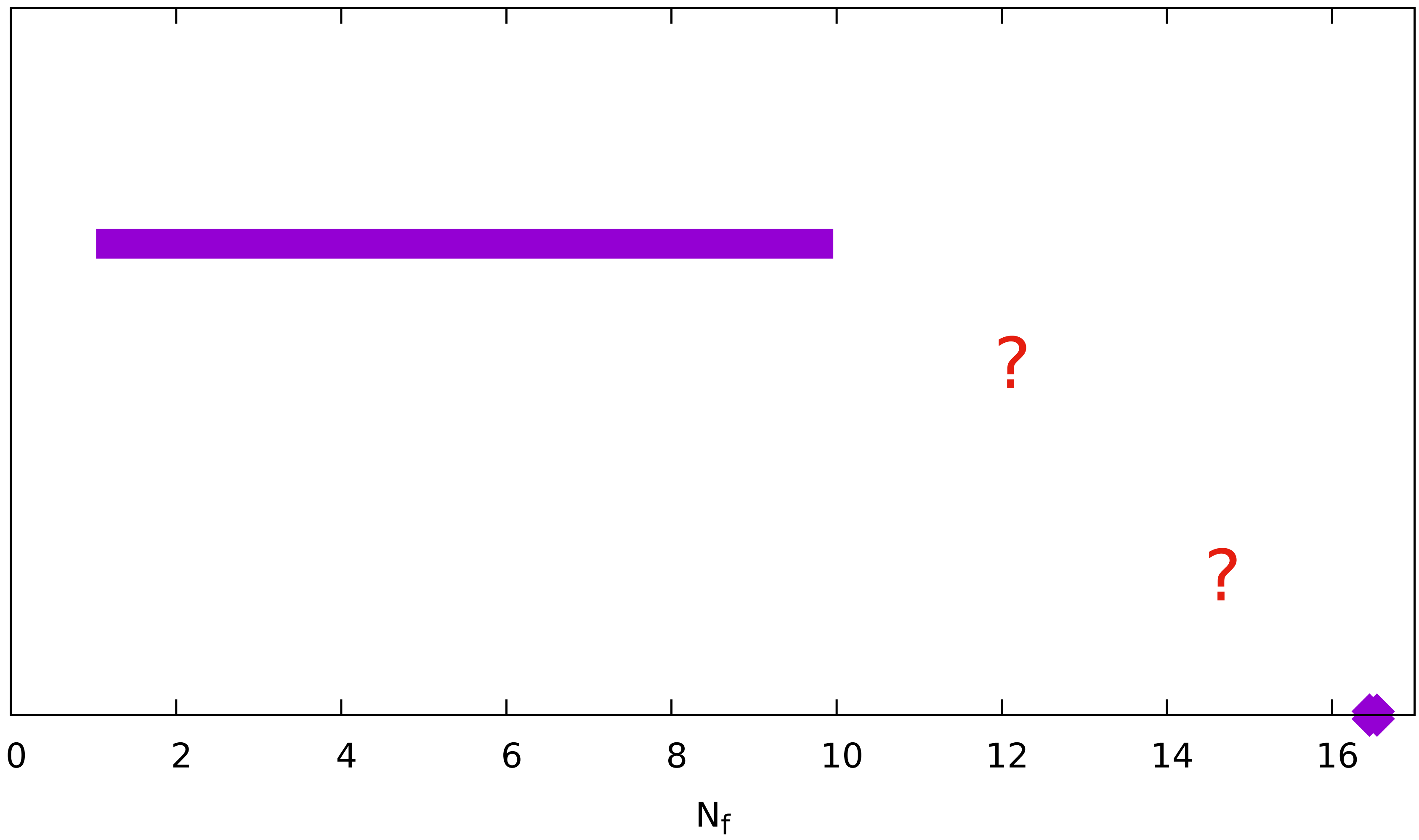
Setup

High N_f

- $N_f = 16.5$, free theory
- $m_V = 2m$
- $f_{PS,V} = 0$
- $f_{PS,V}/m_V = 0$

Something happens between $N_f = 10$ (non-zero ratio) and $N_f = 16.5$ (zero ratio)

Cartoon



Goals

Calculate $f_{PS,V}$ and m_V in perturbation theory

See how far down we can go from $N_f = 16.5$

Hopefully match with highest $N_f = 10$ from the lattice studies

Outline

- Pertubative calculation schematically, Banks-Zaks expansion
- Bound states in perturbation theory
- NRQCD and pNRQCD
- NLO, NNLO, N^3LO results
- Matching between low N_f and high N_f

Perturbative calculation schematically

In QCD running scale μ, $a(\mu) = \frac{g^2(\mu)}{16\pi^2}$

Starting with CFT + mass deformation choose $\mu = m$, $a(m)$

(p)NRQCD will give

$$f_{PS,V} = m\, a^{3/2}(m)\,(b_0 + b_1\, a(m) + \ldots)$$

$$m_V = m(c_0 + c_1\, a^2(m) + \ldots)$$

Here $\ldots$ contains $\log(a)$ too, coefficients depend on N_f

Perturbative calculation schematically

Ratio, m drops out

Take chiral limit $m \to 0$, $a(m) \to a_*$ fixed point

$$\frac{f_{PS,V}}{m_V} = a_*^{3/2}(d_0 + d_1\, a_* + d_2\, a_*^2 + \ldots)$$

Here ... contains $\log(a_*)$ too, coefficients depend on N_f

Banks-Zaks expansion of a_*

$\varepsilon = 16.5 - N_f$ distance from upper end of conformal window

Use 5-loop β-function to calculate a_* : $\quad \beta(a_*) = 0$

Expand a_* in ε

$$a_* = \varepsilon\,(e_0 + e_1\,\varepsilon + e_2\,\varepsilon^2 + e_3\,\varepsilon^3 + \ldots)$$

Now no logs, coefficients are constants

Finally Banks-Zaks expansion of ratio

a_* expanded in ε and $f_{PS,V}/m_V$ expanded in a_*

$$\frac{f_{PS,V}}{m_V} = \varepsilon^{3/2}(h_0 + h_1\,\varepsilon + h_2\,\varepsilon^2 + \ldots)$$

Here ... contains $\log(\varepsilon)$ too, coefficients are constants

These will be our main results

Bound states in perturbation theory

$f_{PS,V}$ and m_V are properties of bound states (mesons)

Leading order: 2 non-interacting fermions

$$m_V = 2m$$
$$f_{PS,V} = 0$$

Valid at $N_f = 16.5$

Bound states in perturbation theory

First correction: 1-gluon exchange, Coulomb potential

$$V^{(0)}(r) = -4\pi\, C_F\, \frac{a}{r}$$

Non-relativistic Schroedinger equation

$$m_V = 2m(1 - 2C_F^2\pi^2a^2 + O(a^3))$$

From wave function at origin $f_{PS,V} \sim |\psi(0)|$

$$f_{PS,V} = ma^{3/2}\pi\sqrt{8NC_F^3}(1 + O(a))$$

Bound states in perturbation theory, (p)NRQCD

Systematic improvement

Integrate out heavy fermions $\rightarrow$ effective theory

Fermions are slow moving $v \ll 1$ and coupling is small $a \ll 1$

Non-relativistic setup, relativistic effects from corrections

(p)NRQCD

3 scales

- m fermion mass (hard scale)

- mv typical fermion momentum in meson rest frame (soft scale)

- mv^2 typical kinetic energy

In QCD: Λ_{QCD} too, but not for us (CFT)

Separation $$mv^2 \ll mv \ll m$$

NRQCD

Developed for heavy-heavy mesons in QCD, $\Upsilon, J/\Psi, \ldots$

Low energy properties of heavy mesons

Analogous but much simpler: (p)NRQED and positronium

In NRQCD language: n_1 light quarks and n_2 heavy quarks

Integrate out n_2 heavy quarks, n_1 light quarks stay

For us: $n_1 = 0, \;\; n_2 = N_f$ because start from CFT

NRQCD

- Define fields for low-energy excitations
- Impose symmetries
- Specify how accurate the effective theory should be
- Find most general Lagrangian for required accuracy
- Find coefficients from matching original and effective theory

NRQCD

Heavy fermion and anti-fermion, Pauli spinor ψ and χ

$$\mathcal{L}^{(0)} = \psi^\dagger \left(iD_t + \frac{\mathbf{D}^2}{2m} \right) \psi + \chi^\dagger \left(iD_t - \frac{\mathbf{D}^2}{2m} \right) \chi$$

Higher orders

$$\mathcal{L}^{(1)} = \frac{c_1}{8m^3} \psi^\dagger (\mathbf{D}^2)^2 \psi + \frac{c_2}{8m^2} \psi^\dagger \left(\mathbf{D} g\mathbf{E} - g\mathbf{E}\mathbf{D} \right) \psi +$$

$$+ \frac{c_3}{8m^2} \psi^\dagger \left(i\mathbf{D} \times g\mathbf{E} - g\mathbf{E} \times \mathbf{D} \right) \sigma \psi + \frac{c_4}{2m} \psi^\dagger g\mathbf{B}\sigma\psi + (\psi \to \chi)$$

Even more at further orders ...

Short distance coefficients $c_{1,2,3,4} = 1 + O(a)$

Calculate these to desired order in a: matching QCD to NRQCD

pNRQCD

Further integrate out soft scale mv

Decay constants from NRQCD matrix elements: long distance matrix elements

Long distance matrix elements from pNRQCD wave functions

Wave functions from Schroedinger equation with $V(r)$

$$V(r) = V^{(0)}(r) + V^{(1)}(r, \partial) + \ldots$$

Expanded in $1/m$, determined by NRQCD - pNRQCD matching pNRQCD

Long distance matrix elements, $O(v)$

Pseudo scalar: $\langle 0|\chi^\dagger\psi|PS\rangle = \sqrt{2N_c}|\Psi_{PS}(0)|$

Vector: $\langle 0|\chi^\dagger\epsilon\sigma\psi|V\rangle = \sqrt{2N_c}|\Psi_V(0)|$

Long distance matrix elements, $O(v^2)$

Pseudo scalar: $\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|PS\rangle = mE_{PS}\sqrt{2N_c}|\Psi_{PS}(0)|$

Vector: $\langle 0|\chi^\dagger\epsilon\sigma\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|V\rangle = mE_V\sqrt{2N_c}|\Psi_V(0)|$

$E_{PS,V} = m_{PS,V} - 2m = O(a^2) < 0$ binding energies

pNRQCD, decay constants

$$f_{PS} = \frac{1}{\sqrt{m_{PS}}}\left(c_p\langle 0|\chi^\dagger\psi|PS\rangle - \frac{d_p}{2m^2}\langle 0|\chi^\dagger(-\frac{i}{2}\overleftrightarrow{\mathbf{D}})^2\psi|PS\rangle\right)$$

$$f_V = \frac{1}{\sqrt{m_V}}\left(c_v\langle 0|\chi^\dagger\epsilon\cdot\sigma\psi|V\rangle - \frac{d_v}{6m^2}\langle 0|\chi^\dagger\epsilon\cdot\sigma(-\frac{i}{2}\overleftrightarrow{\mathbf{D}})^2\psi|V\rangle\right)$$

where $c_v, d_v, c_p, d_p = 1 + O(a)$ matching coefficients

$$f_{PS} = \sqrt{\frac{N_c}{m}}\left[c_p - \left(\frac{c_p}{4} + \frac{d_p}{2}\right)\frac{E_{PS}}{m}\right]|\Psi_{PS}(0)|$$

$$f_V = \sqrt{\frac{N_c}{m}}\left[c_v - \left(\frac{c_v}{4} + \frac{d_v}{6}\right)\frac{E_V}{m}\right]|\Psi_V(0)|$$

Decay constants

N_f-dependence from the matching coefficients

From heavy quark literature:

- NNLO for all ingredients of f_{PS}
- N^3LO for all ingredients of f_V

In QCD: $a(\mu) = \frac{g^2(\mu)}{16\pi^2}$ non-trivial μ-dependence

Starting with CFT + mass deformation, natural choice $\mu = m$

Results, m_V

$$m_V = c_0 m \left(1 + c_2 a^2(m) + c_{30} a^3(m) + c_{31} a^3(m) \log a(m) + O(a^4)\right)$$

$$c_0 = 2$$

$$c_2 = -2 C_F^2 \pi^2$$

$$c_{30} = \frac{4}{9} \pi^2 C_A C_F^2 \left(66 \log(4\pi C_F) - 97\right)$$

$$c_{31} = \frac{88}{3} \pi^2 C_A C_F^2$$

Results, f_V NNLO

$$f_V = b_0^V\, m\, a^{3/2}(m) \left(1 + \sum_{n=1}^{3} \sum_{k=0}^{n} b_{nk}^V\, a^n(m) \log^k a(m) + O(a^4) \right)$$

$$b_0^V = \sqrt{8 N_c C_F^3 \pi}, \qquad b_{10}^V = \frac{161}{6} - \frac{11\pi^2}{3} + 33 \log\left(\frac{3}{16\pi}\right), \qquad b_{11}^V = -33$$

$$b_{20}^V = \left(-\frac{64\pi^2}{27} + \frac{704}{27}\right) N_f + \frac{9781\zeta(3)}{9} - \frac{27\pi^4}{8} + \frac{1126\pi^2}{81} + \frac{9997}{72} +$$

$$+ \frac{1815 \log^2 \pi}{2} + \frac{1815}{2} \log^2\left(\frac{16}{3}\right) + \log\left(\frac{16}{3}\right)\left(-\frac{2581}{2} + \frac{605\pi^2}{3} + 1815 \log(\pi)\right) +$$

$$+ \left(\frac{4325\pi^2}{27} - \frac{2581}{2}\right) \log(\pi) - \frac{256}{81}\pi^2 \log(8) - \frac{1120}{27}\pi^2 \log\left(\frac{8}{3}\right) - \frac{512}{9}\pi^2 \log(2)$$

$$b_{21}^V = \frac{4325\pi^2}{27} - \frac{2581}{2} + 1815 \log\left(\frac{16\pi}{3}\right), \qquad b_{22}^V = \frac{1815}{2}.$$

Results, f_V N^{3}LO

$$f_V = b_0^V \, m \, a^{3/2}(m) \left(1 + \sum_{n=1}^{3} \sum_{k=0}^{n} b_{nk}^V a^n(m) \log^k a(m) + O(a^4) \right)$$

$$b_{30}^V = 0.8198 N_f^2 - 362.7 N_f - 1.0901(1) \times 10^6$$

$$b_{31}^V = -88.42 N_f - 7.7493 \times 10^5$$

$$b_{32}^V = -2.1651 \times 10^5$$

$$b_{33}^V = -2.3292 \times 10^4$$

Part of it numerical only

Results, f_{PS} NNLO

$$f_{PS} = b_0^{PS} m a^{3/2}(m) \left(1 + \sum_{n=1}^{2} \sum_{k=0}^{n} b_{nk}^{PS} a^n(m) \log^k a(m) + O(a^3) \right)$$

$$b_0^{PS} = \sqrt{8 N_c C_F^3 \pi}, \quad b_{10}^{PS} = \frac{59}{2} - \frac{11\pi^2}{3} + 33 \log\left(\frac{3}{16\pi}\right), \quad b_{11}^{PS} = -33$$

$$b_{20}^{PS} = N_f \left(-\frac{32\pi^2}{9} + \frac{344}{9} \right) + 961\zeta(3) - \frac{27\pi^4}{8} + \frac{1310\pi^2}{27} + \frac{23053}{72} +$$

$$+ \frac{1815 \log^2 \pi}{2} + \frac{1815}{2} \log^2\left(\frac{16}{3}\right) + \log\left(\frac{16}{3}\right) \left(-\frac{2757}{2} + \frac{1271\pi^2}{9} + 1815 \log \pi \right) +$$

$$+ \left(\frac{1271\pi^2}{9} - \frac{2757}{2} \right) \log \pi - \frac{272}{9} \pi^2 \log 2$$

$$b_{21}^{PS} = \frac{1271\pi^2}{9} - \frac{2757}{2} + \frac{1815}{2} \log\left(\frac{256\pi^2}{9}\right), \qquad b_{22}^{PS} = \frac{1815}{2}.$$

Banks-Zaks expansion

Replace N_f-dependence by $\varepsilon = 16.5 - N_f$

Expand a_* in ε from 5-loop β-function

Now both ratios are just expanded in ε

Main result, Banks-Zaks expansion of ratios

$$\frac{f_V}{m_V} = \varepsilon^{3/2} C_0 \left(1 + \sum_{n=1}^{3} \sum_{k=0}^{n} C_{nk}\, \varepsilon^n \log^k \varepsilon + O(\varepsilon^4) \right)$$

$$C_0 = 0.005826678$$

$$C_{10} = 0.4487893 \quad C_{11} = -0.2056075 \quad C_{20} = 0.2444502$$

$$C_{21} = -0.1624891 \quad C_{22} = 0.03522870 \quad C_{30} = 0.10604(3)$$

$$C_{31} = -0.1128420 \quad C_{32} = 0.03695458 \quad C_{33} = -0.005633665$$

Main result, Banks-Zaks expansion of ratios

$$\frac{f_{PS}}{m_V} = \varepsilon^{3/2} C_0 \left(1 + \sum_{n=1}^{2} \sum_{k=0}^{n} D_{nk}\, \varepsilon^n \log^k \varepsilon + O(\varepsilon^3) \right)$$

$$D_{10} = 0.4654041 \quad D_{11} = -0.2056075$$

$$D_{20} = 0.2845697 \quad D_{21} = -0.1737620 \quad D_{22} = 0.03528692$$

Notes

- Coefficients do not blow up (unlike in terms of a of $f_{V,PS}, m_V$)

- Coefficients are scheme independent

- Many subtle details left out

- Renormalization (we use $\overline{\text{MS}}$)

- Arbitrary scale independence and cancellations ...

f_V/m_V

N^{3}LO perturbative result

Direct lattice results only for f_{PS}

Use KSRF relation to extract f_V

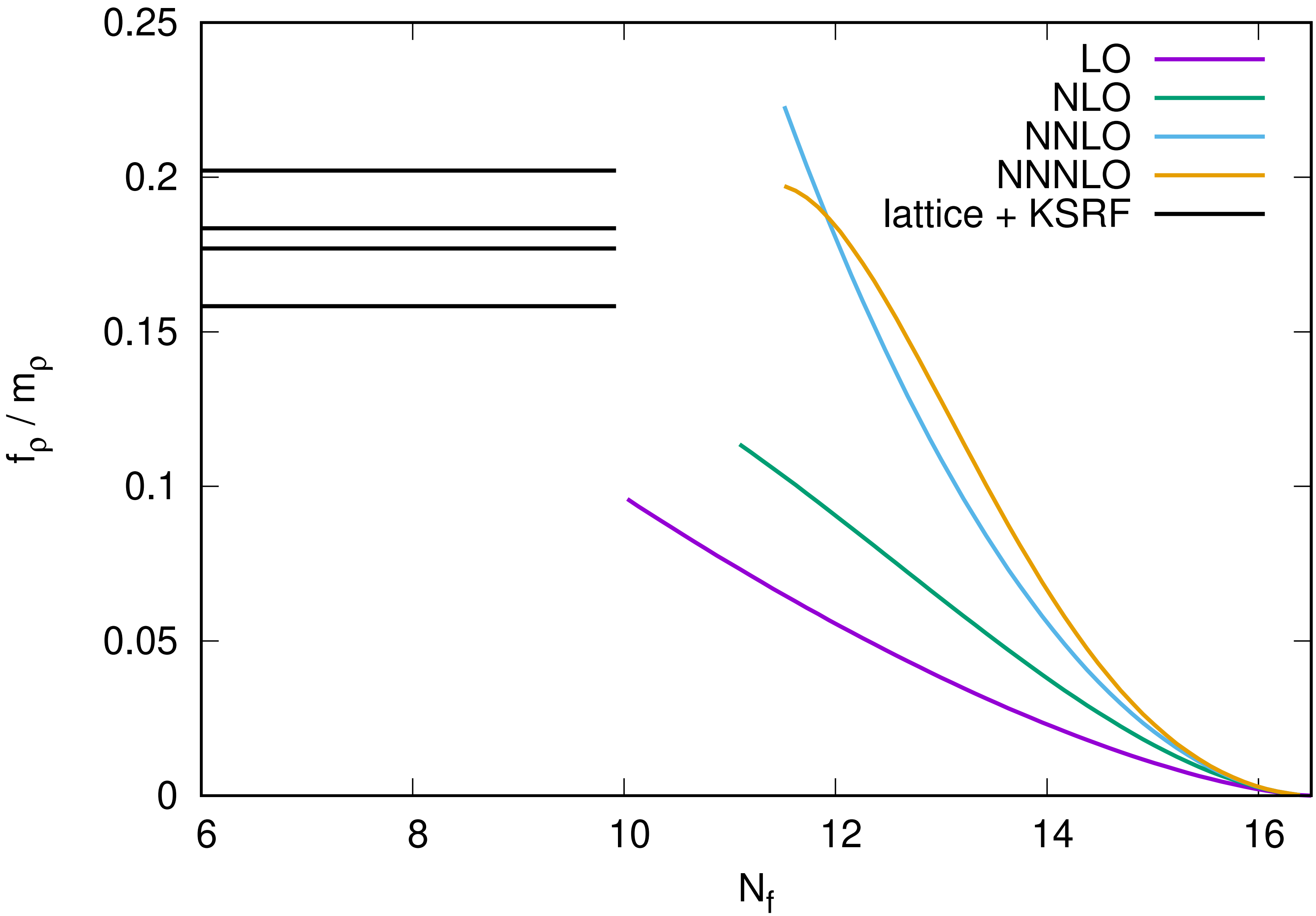
$$f_V = \sqrt{2} f_{PS}$$

From vector meson dominance / universality

Proven in SQCD, correct to about 12% in QCD

Conservatively assign 12% uncertainty

Main result - f_V/m_V

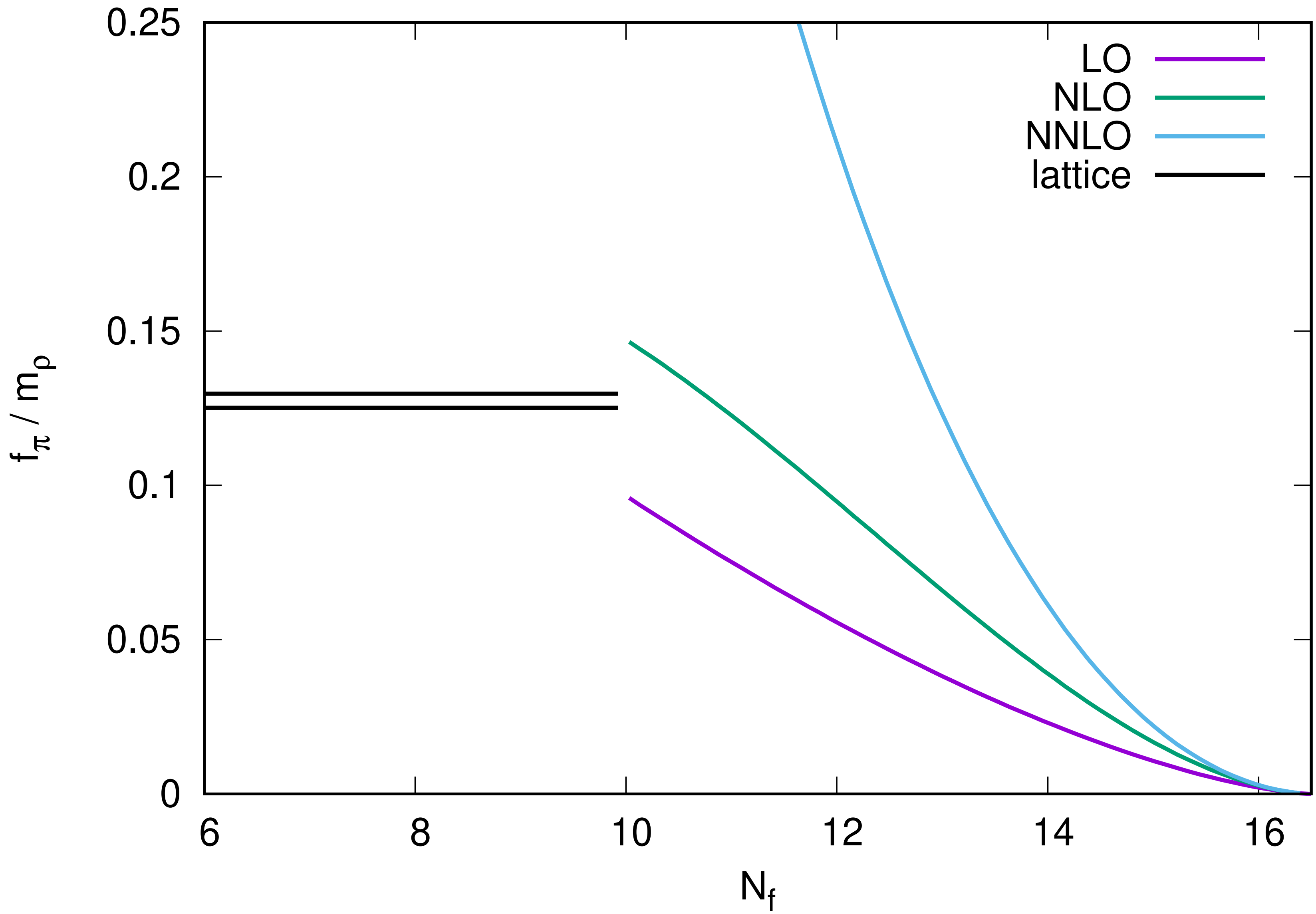


Main result - f_V/m_V

Important observations

- NNLO and N^{3}LO almost the same down to $N_f = 12$
- N^{3}LO matches at $N_f = 12$ last non-pertubative point $N_f = 10$
- Quantitatively
  - $N_f = 12.00(4)$ NNLO
  - $N_f = 12.08(6)$ N^{3}LO
  - $N_f = 12.0(5)$ N^{3}LO + KSRF

Main result - f_{PS}/m_V



Main result - f_{PS}/m_V

Important observations

- N^3LO not available
- Assume similar to f_V/m_V
- Match seems to be around $N_f = 13$

Conclusions

- Perturbation theory perhaps reliable down to $N_f = 12$
- If monotonous N_f-dependence assumed, constrain N_f^*
- $N_f^* \simeq 12$ **and** $N_f^* \simeq 13$ **from the two ratios**
- In any case: abrupt change in ratios at these N_f
- Our method combines perturbative and non-perturbative input

Improvements for the future

- N^{3}LO calculation of f_{PS} (very difficult)
- Direct f_V lattice calculation for $N_f \leq 10$ (probably doable)
- Perhaps $N_f = 11, 12$ lattice calculation (costly)
- N^{4}LO: 6-loop β-function would be needed (not any time soon)

Thank you for your attention!