Trans-series from condensates

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WORK WITH

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Contents

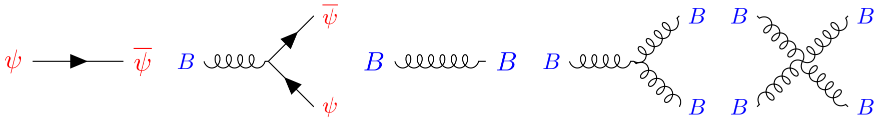
Review on QCD, the Operator Product Expansion and trans-series

QCD: perturbative vs non-perturbative

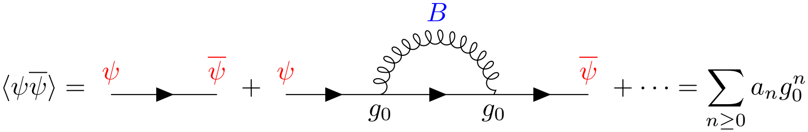
- (Massless) QCD Lagrangian:

$$\mathcal{L} = \overline{\psi}\mathrm{i}\gamma^\mu D_\mu \psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a = \underbrace{\overline{\psi}\mathrm{i}\gamma^\mu \partial_\mu \psi}_{\text{quark free propagator}} + \underbrace{g_0 \frac{\lambda^a}{2} \overline{\psi}\gamma^\mu \psi B^a_\mu}_{\text{quark-gluon interaction}}$$

$$\underbrace{- \frac{1}{4} \left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g_0 f^{abc} B^b_\mu B^c_\nu\right) \left(\partial^\mu B^\nu_a - \partial^\nu B^\mu_a + g_0 f^{abc} B^\mu_b B^\nu_c\right)}_{\text{after expanding: gluon free propagator + gluon self-interactions}}$$



- Let's compute the propagator of a quark in perturbation theory:



QCD: perturbative vs non-perturbative

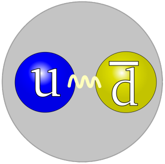
QCD is asymptotically free ($\Lambda_{\text{QCD}} = 332 \pm 17$ MeV):

$$g_0 \xrightarrow{\text{Renormalization}} g(p^2) \approx \frac{1}{\beta_0 \log(p^2/\Lambda_{\text{QCD}}^2)} \xrightarrow{p^2 \to \infty} 0$$

bare coupling — renormalized coupling — small coupling at high energies

Perturbation theory valid at large energies (e.g. colliders). But fails to describe QCD at low energies (NON-perturbative physics):

- QCD has a mass gap: Observable particles can't have arbitrarily low energies. The lowest mass of observable particles is $m \approx 135$ MeV (pions).



- Confinement: Quarks and gluons cannot be observed in isolation.
- Chiral symmetry breaking: Generates masses of hadrons far above the masses of the quarks and makes pions exceptionally light.

Operator Product Expansion (OPE): a trans-series with condensates

How can we extract non-perturbative information from the Lagrangian?

- Do lattice QCD with the help of computers.
- Extend our perturbative computation to a trans-series:

$$\int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}px} \langle \psi(x) \overline{\psi}(0) \rangle = \underbrace{\sum_{n\geq 0} a_n g(p)^n}_{\text{perturbation theory}} + \underbrace{\mathrm{e}^{-1/g(p)} \sum_{n\geq 0} b_n g(p)^n + \mathcal{O}(\mathrm{e}^{-2/g(p)})}_{\text{non-perturbative corrections}}$$

Trans-series: prefered by mathematicians (resurgence)

$\downarrow\uparrow$ equivalent objects

OPE: prefered by physicists (phenomenology)

Operator Product Expansion (OPE): a trans-series with condensates

- The two-point correlator of any local operator can be written as an expansion in other local operators:

$$\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \langle \psi(x)\overline{\psi}(0)\rangle = \sum_i C_i(p)\langle O_i(0)\rangle, \quad p^2 \to \infty,$$

  $C_i(x)$ are the Wilson coefficients,
  $O_i(0)$ are local operators (local = all fields evaluated at the same point):

$$O_i(0) = \prod_i \partial_i \prod_j \psi_j(0) \prod_k B_k(0)$$

  $\langle O_i(0)\rangle$ are called the (vacuum) condensates.

Important condensates in QCD, used in many phenomenological applications (SVZ sum rules):

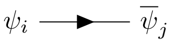
- Identity $\langle I\rangle = 1$: Its Wilson coefficients give rise to perturbation theory.
- Quark condensate $\langle m\overline{\psi}\psi\rangle$: Breaks chiral symmetry, explaining masses of mesons, e.g. $M_\pi^2 \propto (m_u + m_d)\langle\overline{\psi}\psi\rangle$.
- Gluon condensate $\langle G^a_{\mu\nu}G_a^{\mu\nu}\rangle$.

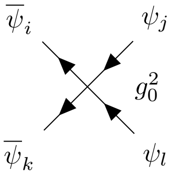
QCD is too hard. Let's consider a toy model

The $\mathrm{U}(N)$ Gross–Neveu model

- A QFT in $1+1$ dimensions: N fermions $\boldsymbol{\psi}(x) = \big(\psi_1(x), \ldots, \psi_N(x)\big)$ with a 4-vertex interaction:

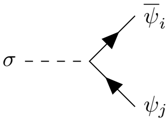
$$\mathcal{L} = \mathrm{i}\overline{\boldsymbol{\psi}} \cdot \gamma^\mu \partial_\mu \boldsymbol{\psi} + \frac{g_0^2}{2}(\overline{\boldsymbol{\psi}} \cdot \boldsymbol{\psi})^2$$





- Better to consider an equivalent Lagrangian with a bosonic auxiliary field σ:

$$\mathcal{L}_\sigma = \mathrm{i}\overline{\boldsymbol{\psi}} \cdot \gamma^\mu \partial_\mu \boldsymbol{\psi} - \frac{1}{2}\sigma^2 + g_0 \sigma \overline{\boldsymbol{\psi}} \cdot \boldsymbol{\psi},$$



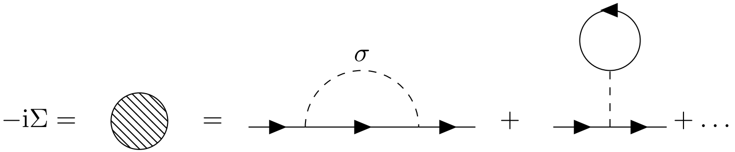
- Easy to check that both Lagrangians are equivalent.

The fermion two-point correlator and the self-energy

- Let's compute the two-point correlator of the fermions ($\not{p} = \gamma^\mu p_\mu$):

$$\int \mathrm{d}^2x\, \mathrm{e}^{\mathrm{i}px} \langle \psi(x)\overline{\psi}(0)\rangle = \frac{\mathrm{i}}{\not{p} - \Sigma} = \frac{\mathrm{i}}{\not{p}} + \frac{\mathrm{i}}{\not{p}}(-\mathrm{i}\Sigma)\frac{\mathrm{i}}{\not{p}} + \frac{\mathrm{i}}{\not{p}}(-\mathrm{i}\Sigma)\frac{\mathrm{i}}{\not{p}}(-\mathrm{i}\Sigma)\frac{\mathrm{i}}{\not{p}} + \dots$$

- The blob is the so-called self-energy of the fermion:



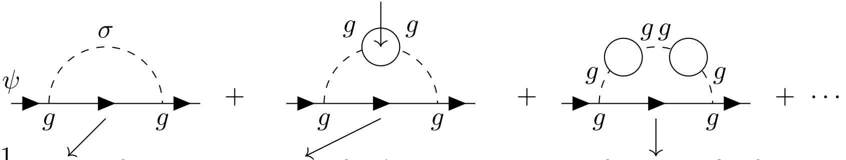
1-particle irreducible diagrams

NOT irreducible

$\mathrm{U}(N)$ Gross–Neveu model at large N

- 2-point correlator $\langle \psi(x)\overline{\psi}(0)\rangle$ in the large N limit:

N fermions contributing to the loop



$$\frac{1}{N}\left[\color{red}{a_{0,0}}Ng^2 + \big(a_{1,0}N + \color{red}{a_{1,1}}N^2\big)g^4 + \big(a_{2,0}N + a_{2,1}N^2 + \color{red}{a_{2,2}}N^3\big)g^6 + \cdots\right]$$

- Can be computed exactly in dimensional regularization:

$$\begin{aligned}\Sigma &= \frac{\not{p}}{N}\int_0^\infty \mathrm{e}^{-y/\lambda}\frac{1}{4-2y}\mathrm{d}y \\ &= \frac{\not{p}}{N}\left[\frac{\lambda}{4} + \frac{\lambda^2}{8} + \frac{\lambda^3}{8} + \frac{3\lambda^4}{16} + \mathcal{O}(\lambda^5)\right] + \mathcal{O}(1/N^2) + \mathcal{O}(\mathrm{e}^{-2/\lambda}),\end{aligned}$$

[Campostrini, Rossi 1992] where $\lambda = \dfrac{Ng^2}{\pi}$ ('t Hooft coupling).

- But we are still missing all non-perturbative corrections!

$\mathrm{U}(N)$ Gross–Neveu model at large N: an alternative way

- There is a second way to compute the self-energy, which incorporates non-perturbative corrections, but only suited for special (toy) models.
- Recalling the Lagrangian:

$$\mathcal{L}_\sigma = \mathrm{i}\overline{\boldsymbol{\psi}} \cdot \gamma^\mu \partial_\mu \boldsymbol{\psi} - \frac{1}{2}\sigma^2 + g_0 \sigma \overline{\boldsymbol{\psi}} \cdot \boldsymbol{\psi}$$

Realize that it is quadratic in the fermions, so they can be integrated out. This yields an effective action:

$$S_{\text{eff}} = \int \mathrm{d}^2 x \left[-\frac{\sigma^2}{2g_0^2} - \mathrm{i}N \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \operatorname{Tr} \log(\not{p} - \sigma) \right].$$

- σ must satisfy the equations of motion:

$$\left.\frac{\delta S_{\text{eff}}}{\delta \sigma}\right|_{\sigma=\sigma_c} = 0 \quad \Longrightarrow \quad \frac{1}{N g_0^2} = \frac{1}{\sigma_c} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \operatorname{Tr}\left[\frac{\mathrm{i}}{\not{p} - \sigma_c}\right]$$

- Solving for σ_c and computing the integral in dimensional regularization:

$\sigma_c \equiv m = p\, \mathrm{e}^{-1/\lambda}$ is a non-perturbative parameter. $\quad \left(\lambda = \frac{N g^2}{\pi}\right)$

$\mathrm{U}(N)$ Gross–Neveu model at large N: an alternative way

- Using the non-perturbative propagator for σ, we obtain the non-perturbative self-energy of the fermions (at large N) [Campostrini, Rossi 1992]

$$\Sigma(p) = m + \frac{1}{N}\left(\not{p}\Sigma_p(p^2) + m\Sigma_m(p^2)\right) + \mathcal{O}(1/N^2) + \cancel{\mathcal{O}(\mathrm{e}^{-1/\lambda})}$$

$$\Sigma_p(p^2) = \frac{1}{4p^2}\int_0^\infty \mathrm{d}k^2 \left[\xi \log\left(\frac{\xi+1}{\xi-1}\right)\right]^{-1} \left[1 - \frac{p^2+k^2+m^2}{\sqrt{(p^2-k^2-m^2)^2+4p^2k^2}}\right],$$

$\Sigma_m(p^2) =$ A similar expression . . .

with $\xi = \sqrt{1-4m^2/k^2}$.

- Fermions behave as if they had mass m ("propagator" $= \dfrac{\mathrm{i}}{\not{p}-\Sigma} \approx \dfrac{\mathrm{i}}{\not{p}-m}$).

  m is the mass gap of the theory.

- The integral can be computed numerically for different values of p^2/m^2, but it is more instructive to compute its trans-series expansion for small values of

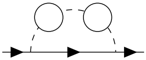
$$\lambda = \frac{2}{\log(p^2/m^2)}, \quad (p^2 \to \infty), \quad \left(m = p\,\mathrm{e}^{-1/\lambda}\right)$$

$\mathrm{U}(N)$ Gross–Neveu model at large N: an alternative way

- Extracting the trans-series would be impossible if not for a trick found in [Beneke, Braun, Kivel 1998]:

$$\Sigma(p) = m + \frac{1}{N}\left(\not{p}\,\Sigma_p(p^2) + m\Sigma_m(p^2)\right) + \mathcal{O}(1/N^2)$$

pert. exp.



$$\Sigma_p(p^2) = \overbrace{\sum_{k\geq 1}\frac{(k-1)!}{2^{k+1}}\lambda^k}^{\text{pert. exp.}} \underbrace{+\,\mathrm{e}^{-2/\lambda}\left[-\log(\lambda) + \gamma_E + \log(2) \pm \mathrm{i}\frac{\pi}{2} - \frac{\lambda}{4} + \frac{\lambda^2}{8} + \sum_{k\geq 1}\frac{(2k+1)!}{2^{2k+2}}\zeta(2k+1)\lambda^{2k+2}\right] + \mathcal{O}(\mathrm{e}^{-4/\lambda})}_{\text{New non-perturbative terms}}$$

$$\Sigma_m(p^2) = -\frac{1}{2}\log(2) - \frac{\gamma_E}{2} + \frac{1}{2} + \frac{1}{2}\log(\lambda) + \sum_{k\geq 1}\frac{(2k)!}{2^{2k+1}}\zeta(2k+1)\lambda^{2k+1} + \mathcal{O}(\mathrm{e}^{-2/\lambda})$$

$m\Sigma_m(p^2)$ gives non-perturbative corrections starting at $m = p\,\mathrm{e}^{-1/\lambda}$.

The Gross–Neveu OPE

- The exact large N result can only be computed because the Lagrangian is quadratic in the fermions.
- This will not work in QCD. What can we do then?

An OPE computation with condensates

- Goal: Compute the Wilson coefficients in the Gross–Neveu OPE

$$\int \mathrm{d}^2x\, \mathrm{e}^{\mathrm{i}px} \langle \psi(x)\overline{\psi}(0)\rangle = C_I(p) + C_{\overline{\psi}\psi}(p)\langle \overline{\psi}(0)\psi(0)\rangle$$

$$+ C_{\overline{\psi}\partial\!\!\!/\psi}(p)\langle \overline{\psi}(0)\partial\!\!\!/\psi(0)\rangle + C_{(\overline{\psi}\psi)^2}(p)\langle (\overline{\psi}(0)\psi(0))^2\rangle + \boxed{\text{operators of higher dim}}$$

Classical dimension of the operators (ψ has dimension $1/2$):

- $\overline{\psi}\psi$ has dimension 1 $\Rightarrow \langle \overline{\psi}\psi\rangle \propto m = p\,\mathrm{e}^{-1/\lambda}$ (recall the term $m\Sigma_m$ in the self-energy).
- $\overline{\psi}\partial\!\!\!/\psi$ and $(\overline{\psi}\psi)^2$ have dimension 2 $\Rightarrow \langle O\rangle \propto m^2 = p^2\mathrm{e}^{-2/\lambda}$.

The Gross–Neveu OPE

- How can we compute Wilson coefficients?

$$S = \int \mathrm{d}^2 x \left[\underbrace{\mathrm{i}\overline{\boldsymbol{\psi}} \cdot \gamma^\mu \partial_\mu \boldsymbol{\psi} - \frac{1}{2}\sigma^2}_{S_{\text{free}}} + \underbrace{g_0 \sigma \overline{\boldsymbol{\psi}} \cdot \boldsymbol{\psi}}_{g_0 S_{\text{int}}} \right]$$

- Let's recall standard perturbation theory . . .

$$\begin{aligned} \langle \psi(x)\overline{\psi}(0) \rangle &= \int \mathcal{D}\psi \mathcal{D}\sigma \, \mathrm{e}^{\mathrm{i}S_{\text{free}}} \mathrm{e}^{\mathrm{i}g_0 S_{\text{int}}} \psi(x)\overline{\psi}(0) \\ &= \int \mathcal{D}\psi \mathcal{D}\sigma \, \mathrm{e}^{\mathrm{i}S_{\text{free}}} \left[1 + \cancel{g_0 \frac{\mathrm{i}S_{\text{int}}}{1!}} + g_0^2 \frac{(\mathrm{i}S_{\text{int}})^2}{2!} + \ldots \right] \psi(x)\overline{\psi}(0) \end{aligned}$$

- Equivalently, we can use Wick's theorem:

$$\langle \Omega | \mathcal{T} \psi(x)\overline{\psi}(0) | \Omega \rangle \equiv \int \mathcal{D}\psi \mathcal{D}\sigma \, \mathrm{e}^{\mathrm{i}S_{\text{free}}} \psi(x)\overline{\psi}(0)$$

$$\psi(x)\overline{\psi}(0) \xrightarrow{\text{Fourier}} \frac{\mathrm{i}}{\not{p}}$$

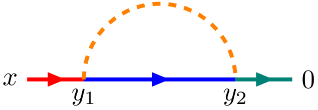
$$x \longrightarrow 0$$

$\mathcal{T}$ denotes the time order product.

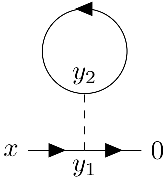
At order g_0^2, Wick contraction of fields give two different diagrams

$$\frac{g_0^2}{2!}\int \mathrm{d}^2 y_1 \mathrm{d}^2 y_2 \langle\Omega|\mathcal{T}\underbrace{\sigma(y_1)\overline{\psi}(y_1)\psi(y_1)}_{S_{\text{int}}}\underbrace{\sigma(y_2)\overline{\psi}(y_2)\psi(y_2)}_{S_{\text{int}}}\psi(x)\overline{\psi}(0)|\Omega\rangle$$

- $\psi(x)\overline{\psi}(y_1)\ \psi(y_1)\overline{\psi}(y_2)\ \psi(y_2)\overline{\psi}(0)\ \sigma(y_1)\sigma(y_2)$



- $\psi(x)\overline{\psi}(y_1)\ \psi(y_1)\overline{\psi}(0)\ \psi(y_2)\overline{\psi}(y_2)\ \sigma(y_1)\sigma(y_2)$



Kind of unlucky, because both diagrams are 0, but . . .

Wick's theorem

- Let us consider an arbitrary field with some indices μ_i at position x_i: $\phi^{\mu_i}(x_i) \equiv \phi_i$.

$$\wick{\c{\phi}_i \c{\phi}_j} = \text{ free propagator}$$

$$:\phi_i\phi_j: = \boxed{\begin{array}{l}\text{normal ordered product (put all annihilation operators}\\ \text{to the right of creation operators)}\end{array}}$$

$$\begin{aligned}\mathcal{T}\phi_1\phi_2\phi_3\phi_4 = {} & \overbrace{\wick{\c{\phi}_1\c{\phi}_2\c{\phi}_3\c{\phi}_4} + \wick{\c{\phi}_1\c{\phi}_3\c{\phi}_2\c{\phi}_4} + \wick{\c{\phi}_1\c{\phi}_4\c{\phi}_2\c{\phi}_3}}^{\text{perturbation theory}} \\ & \overbrace{+ \ \wick{\c{\phi}_1\c{\phi}_2}:\phi_3\phi_4: + \wick{\c{\phi}_1\c{\phi}_3}:\phi_2\phi_4: + \wick{\c{\phi}_1\c{\phi}_4}:\phi_2\phi_3: \\ & + :\phi_1\phi_2:\wick{\c{\phi}_3\c{\phi}_4} + :\phi_1\phi_3:\wick{\c{\phi}_2\c{\phi}_4} + :\phi_1\phi_4:\wick{\c{\phi}_2\c{\phi}_3} \\ & + :\phi_1\phi_2\phi_3\phi_4:}^{\text{normal ordered products of operators annihilate the vacuum}}\end{aligned}$$

- In Grouss–Neveu:

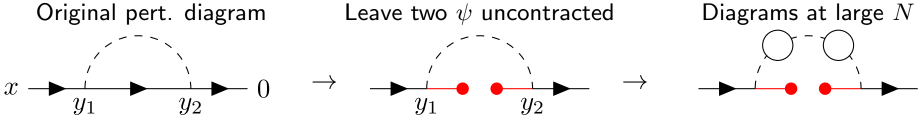
$$\langle\Omega|:\psi(x)\overline{\psi}(y):|\Omega\rangle \text{ is } \begin{cases} = 0 & \text{if using the perturbative vacuum,} \\ \neq 0 & \text{if using the true vacuum} \implies \text{ gives condensates.}\end{cases}$$

Diagrammatic computation of Wilson coefficients

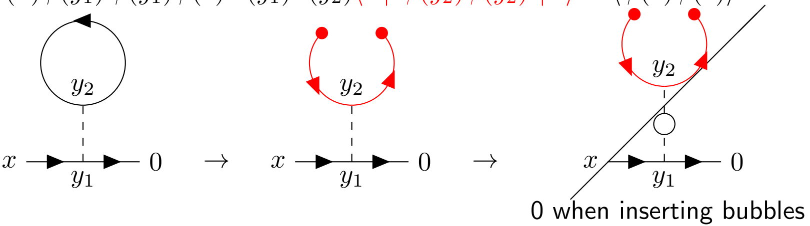
We repeat the perturbative computation, but now we will leave some fields UNCONTRACTED to form the condensates:

- $\overbracket{\psi(x)\overline{\psi}}(y_1)\,\overbracket{\psi(y_2)\overline{\psi}}(0)\,\overbracket{\sigma(y_1)\sigma}(y_2)\,{\color{red}\langle\Omega|{:}\psi(y_1)\overline{\psi}(y_2){:}|\Omega\rangle} \propto$

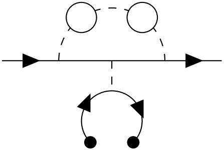
$$\underbrace{\langle\overline{\psi}(0)\psi(0)\rangle}_{\propto \mathrm{e}^{-1/\lambda}} + (\not{y}_1 - \not{y}_2)\underbrace{\langle\overline{\psi}(0)\not{\partial}\psi(0)\rangle}_{\propto \mathrm{e}^{-2/\lambda}} + \mathcal{O}(y_1 - y_2)^2$$



- $\overbracket{\psi(x)\overline{\psi}}(y_1)\,\overbracket{\psi(y_1)\overline{\psi}}(0)\,\overbracket{\sigma(y_1)\sigma}(y_2)\,{\color{red}\langle\Omega|{:}\psi(y_2)\overline{\psi}(y_2){:}|\Omega\rangle} \propto \langle\overline{\psi}(0)\psi(0)\rangle$



Let's put everything together

Still one diagram missing: 

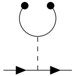
Spend a few weeks computing diagrams in dimensional regularization:

- Diagrams themselves are divergent.
- The bare coupling is divergent (β function).
- The bare fields appearing in the Lagrangian are divergent (Anomalous dimension of the field ψ).
- Local operators are also divergent (Anomalous dimension of the local operators $\overline{\psi}\psi$, $\overline{\psi}\partial\!\!\!/\psi$, $(\overline{\psi}\psi)^2$).

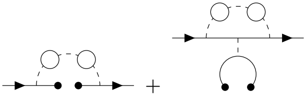
All divergences cancel in the final result.

Everything matches

- After cancellation of divergences among all renormalization constants and diagrams:



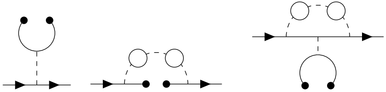
$$C_{\overline{\psi}\psi}\langle[\overline{\psi}\psi]\rangle = \overbrace{m} - \frac{m}{N}\left[1-\frac{\gamma_E}{2}+\log(2) - c_1 + \overbrace{\frac{1}{2}\log(\lambda)}^{\text{Emerges from renormalization constants}} + \underbrace{\sum_{k\geq 1}\frac{(2k)!}{2^{2k+1}}\zeta(2k+1)\lambda^{2k+1}}\right].$$



- The constant c_1 comes from the condensate, which is non-perturbative and cannot be fixed with this diagrammatic method.
- $[\overline{\psi}\psi]$ denotes the renormalized counterpart of the operator.
- Our result matches the exact self-energy at large N.

Conclusions

- Gross–Neveu is a toy model that shares some of the properties of QCD (asymptotic freedom, mass gap and chiral symmetry breaking).
- We computed the fermion two-point correlator and its self-energy, beyond perturbation theory, but at large N, using two methods:

A	B
Integrate out the ψ fields and treat the resulting Lagrangian in the auxiliary field σ. $\mathcal{L} = \mathrm{i}\overline{\boldsymbol{\psi}} \cdot \gamma^{\mu}\partial_{\mu}\boldsymbol{\psi} - \frac{1}{2}\sigma^2 + g_0\sigma\overline{\boldsymbol{\psi}} \cdot \boldsymbol{\psi}$	Do a perturbative computation in the non-perturbative vacuum, including uncontracted terms present in Wick's theorem. 

- Both results match, up to constant terms arising from the condensates.
- This provides a precise test for the validity of the OPE and the method for computing Wilson coefficients.

Many thanks!