Hadronic vacuum polarization contribution to the muon magnetic moment from lattice QCD

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Lepton magnetic moments and BSM physics

Interaction with an external EM field: Dirac eqn

Dirac eqn w/ minimal coupling:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\vec{\alpha}\cdot\left(c\frac{\hbar}{i}\vec{\nabla} - e_\ell\vec{A}\right) + \beta c^2 m_\ell + e_\ell A_0\right]\psi$$

nonrelativistic limit $\downarrow$ (Pauli eqn)

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e_\ell}{c}\vec{A}\right)^2}{2m_\ell} - \underbrace{\frac{e_\ell\hbar}{2m_\ell}\vec{\sigma}\cdot\vec{B}}_{\vec{\mu}_\ell\cdot\vec{B}} + e_\ell A_0\right]\phi$$

with

$$\vec{\mu}_\ell = g_\ell\left(\frac{e_\ell}{2m_\ell}\right)\vec{S}, \qquad \vec{S} = \hbar\frac{\vec{\sigma}}{2}$$

and

$$g_\ell\big|_{\text{Dirac}} = 2$$

Interaction with an external EM field: SM & BSM

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

$$\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m_\ell}\sigma_{\mu\nu}q^\nu F_2(q^2) - \gamma_5\sigma_{\mu\nu}q^\nu F_3(q^2)\right.$$
$$\left. + \gamma_5(q^2\gamma_\mu - 2m_\ell q_\mu)F_4(q^2)\right] u(p)$$

$F_1(q^2)$ $\rightarrow$ Dirac form factor: $F_1(0) = 1$

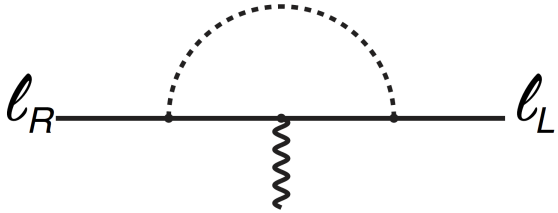
$F_2(q^2)$ $\rightarrow$ Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \dfrac{g_\ell - \overbrace{2}^{g_\ell|_{\text{Dirac}}}}{2}$

$F_3(q^2)$ $\rightarrow$ $\not{P}$, $\not{T}$, electric dipole moment: $F_3(0) = d_\ell/e_\ell$

$F_4(q^2)$ $\rightarrow$ $\not{P}$, anapole moment: $\vec{\sigma}\cdot(\vec{\nabla}\times\vec{B})$

- $F_2(q^2)$ & $F_{3,4}(q^2)$ come from loops but UV finite once theory's couplings are renormalized (in a renormalizable theory)
- a_ℓ dimensionless
  - $\Rightarrow$ corrections including only ℓ and γ are mass independent, i.e. universal
  - $\rightarrow$ contributions from particles w/ $M \gg m_\ell$ are $\propto (m_\ell/M)^{2p} \times \ln^q(m_\ell^2/M^2)$
  - $\rightarrow$ contributions from particles w/ $m \ll m_\ell$ are e.g. $\propto \ln^2(m_\ell^2/m^2)$

Why are a_ℓ special?



$$\longrightarrow \qquad \frac{a_\ell}{2m_\ell} eF^{\mu\nu}[\bar{\ell}_L \sigma_{\mu\nu} \ell_R]$$

- Loop induced $\Rightarrow$ sensitive to new dofs
- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to other measurements: EDMs, $b \to s\ell^+\ell^-$, $\mu \to e\gamma$, $B \to D^{(*)}\ell\nu_\ell$, EW precision observables, LHC direct searches, ...
- In EW theory, only source of chirality flips is $y_\ell \, \bar{\ell}_L H \ell_R$

$$m_\ell = y_\ell \langle H \rangle, \qquad a_\ell^{\text{weak}} \propto \frac{\alpha}{4\pi} \left(\frac{m_\ell}{M_W} \right)^2$$

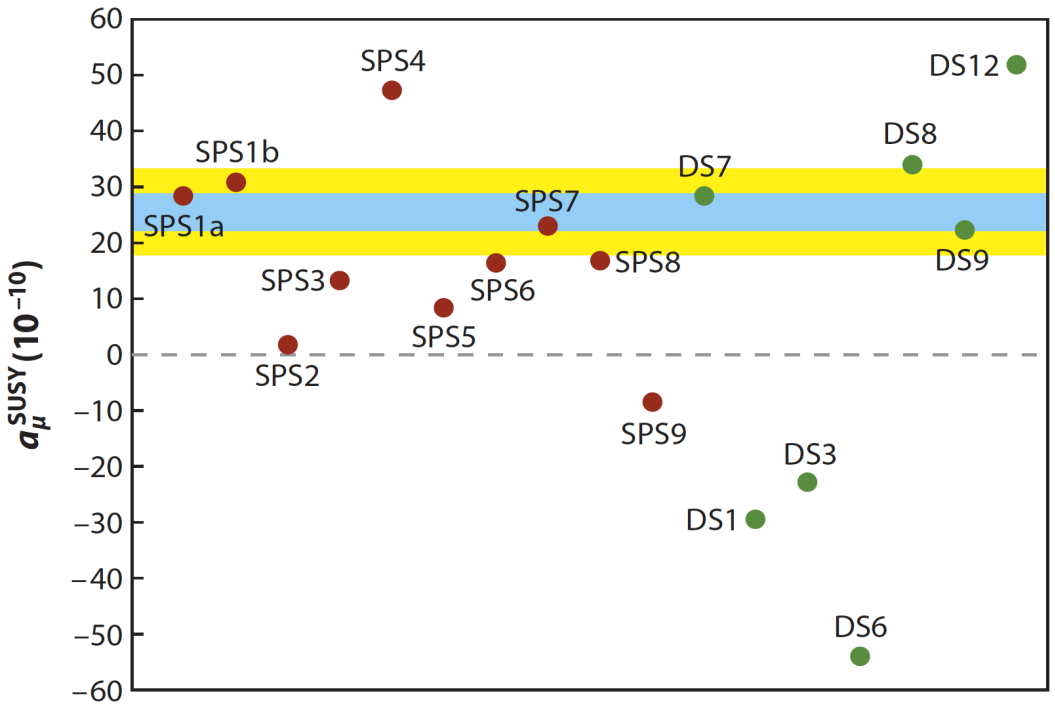
- BSM can be very different

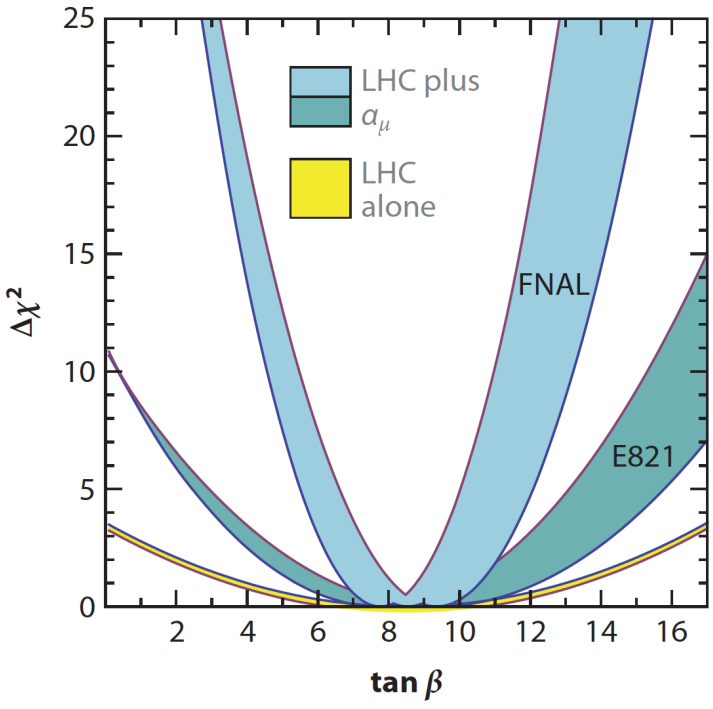
$$a_\ell^{\text{N}\Phi} \propto \left(\frac{\Delta^{\text{N}\Phi} m_\ell}{m_\ell} \right) \left(\frac{m_\ell}{M_{\text{N}\Phi}} \right)^2$$

Why is a_μ special?

$m_e : m_\mu : m_\tau = 0.0005 : 0.106 : 1.777\,\mathrm{GeV}$ $\tau_e : \tau_\mu : \tau_\tau =$ "∞" $: 2.\cdot 10^{-6} : 3.\cdot 10^{-15}$ s

- a_μ is $(m_\mu/m_e)^2 \sim 4. \times 10^4$ times more sensitive to new Φ than a_e
- a_τ is even more sensitive to new Φ, but is too shortly lived





(Miller et al '12)

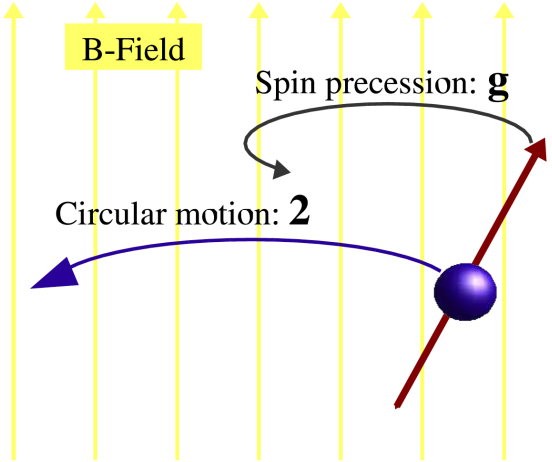
Big question

$$a_\mu^{\text{exp}} = a_\mu^{\text{SM}}?$$

If not, what is the new Φ and can it be seen elswhere?

Experimental measurement of a_μ

Measurement principle for a_μ



$$\vec{\mu}_\mu = 2(1 + a_\mu)\frac{Qe}{2m_\mu}\vec{S}, \qquad \vec{d}_\mu = \eta_\mu \frac{Qe}{2m_\mu c}\vec{S}$$

$$\vec{\omega}_{a\eta} = \vec{\omega}_a + \vec{\omega}_\eta = -\frac{Qe}{m_\mu}\left[a_\mu \vec{B} + \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\frac{\vec{\beta} \times \vec{E}}{c}\right] - \eta_\mu \frac{Qe}{2m_\mu}\left[\frac{\vec{E}}{c} + \vec{\beta} \times \vec{B}\right]$$

- Experiment measures very precisely $\vec{B}$ with $|\vec{B}| \gg |\vec{E}|/c$ &

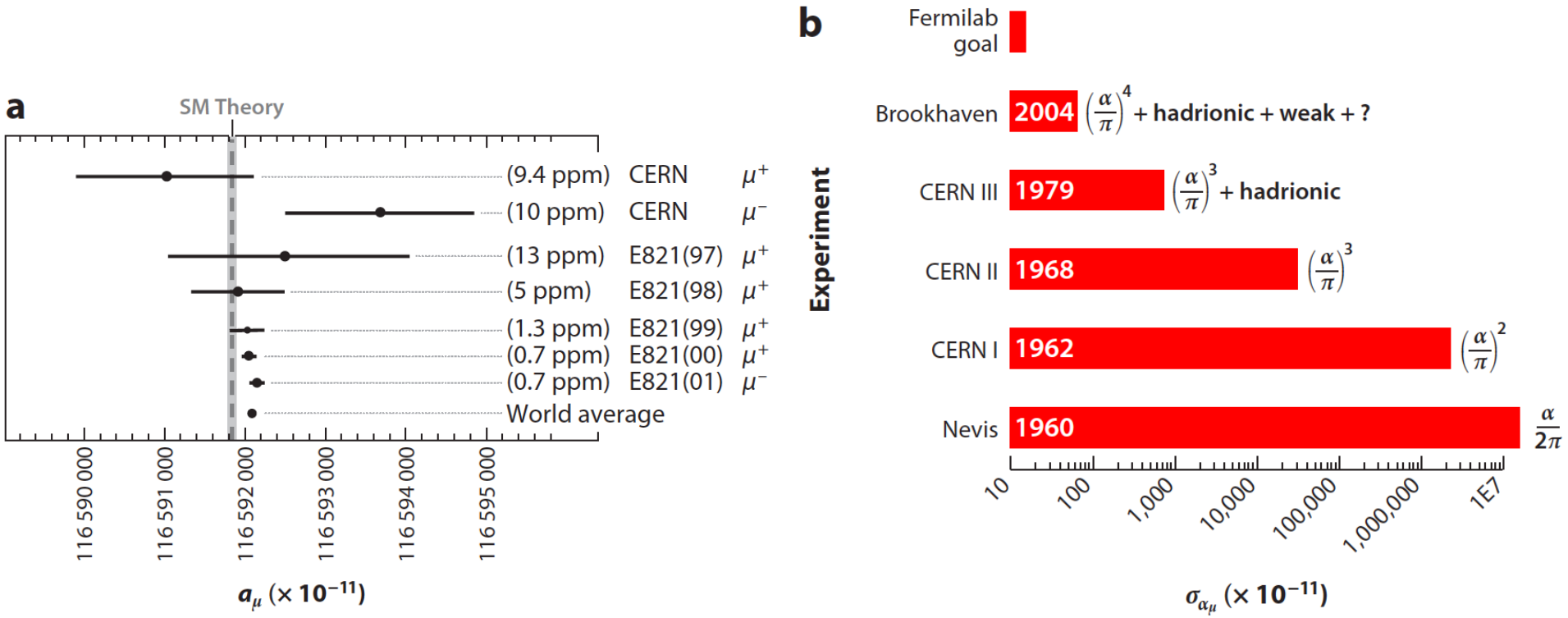
$$\Delta\omega \equiv \omega_S - \omega_C \simeq \sqrt{\omega_a^2 + \omega_\eta^2} \simeq \omega_a$$

since $d_\mu = 0.1(9) \times 10^{-19} e \cdot \text{cm}$ (Benett et al '09)

- Consider either magic $\gamma = 29.3$ (CERN/BNL/Fermilab) or $\vec{E} = 0$ (J-PARC)

$$\rightarrow \Delta\omega \simeq -a_\mu B \frac{Qe}{m_\mu}$$

a_μ experimental summary



(Miller et al '12)

Two new experiments plan to reduce error on a_μ to $\sim$ 0.14 ppm

- New $g-2$ (E989) @ Fermilab: has started taking data fall 2017
- $g-2$/EDM (E34) @ J-PARC: should start taking data $\geq$ 2021

Standard model calculation of a_μ

$$
\begin{aligned}
a_\mu^{\mathrm{SM}} &= a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{had}} + a_\mu^{\mathrm{weak}} \\
&= O\left(\frac{\alpha}{\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{g_2}{4\pi}\right)^2 \left(\frac{m_\mu}{M_W}\right)^2\right) \\
&= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)
\end{aligned}
$$

QED contributions to a_ℓ

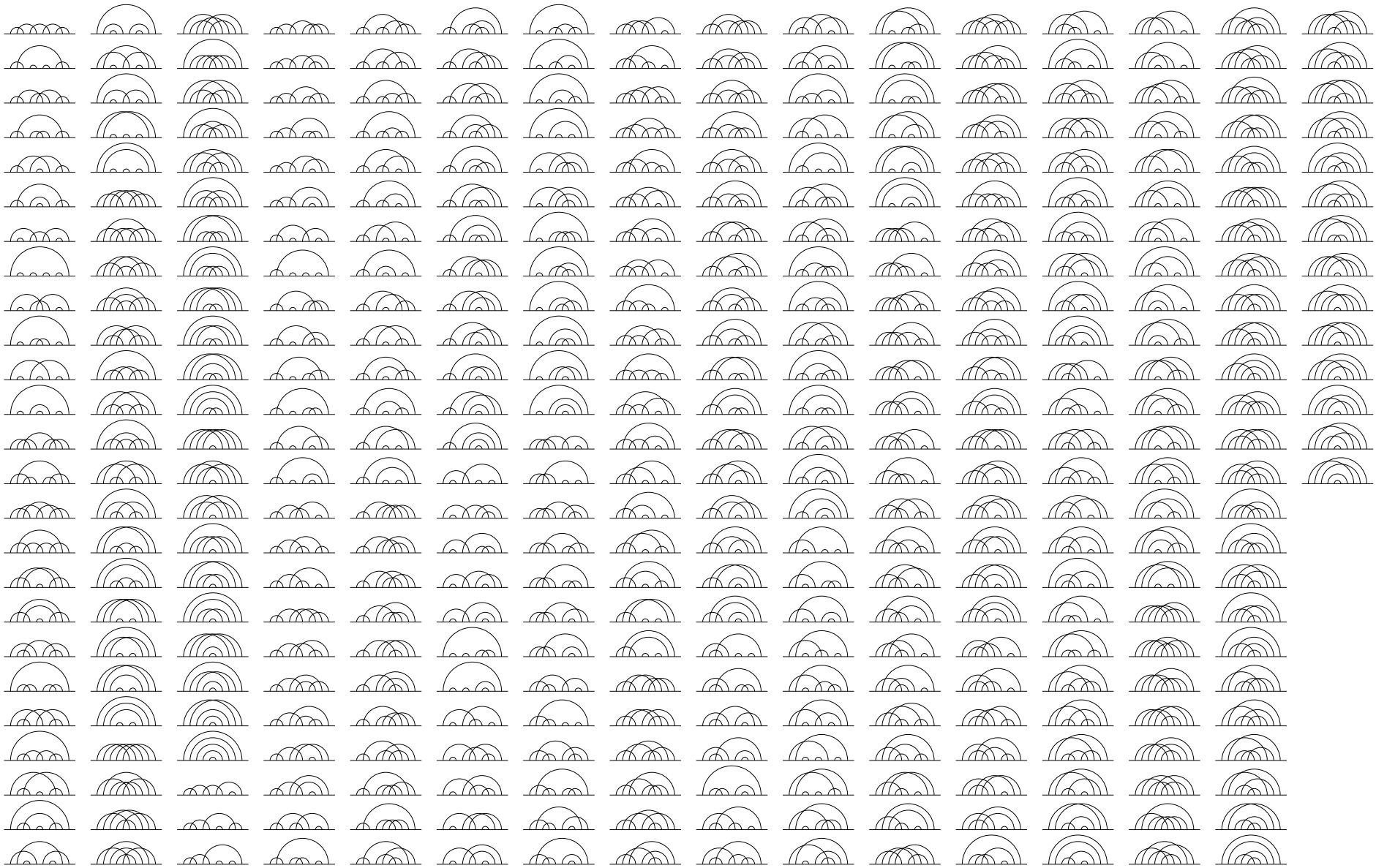
Loops with only photons and leptons

$$a_\ell^{\mathrm{QED}} = C_\ell^{(2)}\left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

$$C_\ell^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_\ell/m_{\ell'}) + A_3^{(2n)}(m_\ell/m_{\ell'}, m_\ell/m_{\ell''})$$

- $A_1^{(2)}$, $A_1^{(4)}$, $A_1^{(6)}$, $A_2^{(4)}$, $A_2^{(6)}$, $A_3^{(6)}$ known analytically (Schwinger '48; Sommerfield '57, '58; Petermann '57; ...)
- $O((\alpha/\pi)^3)$: 72 diagrams (Laporta et al '91, '93, '95, '96; Kinoshita '95)
- $O((\alpha/\pi)^4; (\alpha/\pi)^5)$: 891;12,672 diagrams (Laporta '95; Aguilar et al '08; Aoyama, Kinoshita, Nio '96-'18)
  - Automated generation of diagrams
  - Numerical evaluation of loop integrals
  - Only some diagrams are known analytically

5-loop QED diagrams



(Aoyama et al '15)

QED contribution to a_μ

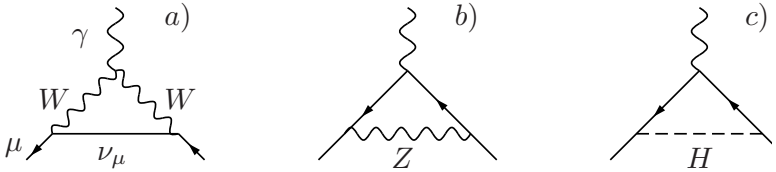
$$a_\mu^{\mathrm{QED}}(Rb) = 1\,165\,847\,189.51(9)_m(19)_{\alpha^4}(7)_{\alpha^5}(77)_{\alpha(Rb)} \times 10^{-12}$$
$$a_\mu^{\mathrm{QED}}(a_e) = 1\,165\,847\,188.41(7)_m(17)_{\alpha^4}(6)_{\alpha^5}(28)_{\alpha(a_e)} \times 10^{-12}$$

(Aoyama et al '12, '18)

$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{QED}} = 737.2(6.3) \times 10^{-10}$$
$$\stackrel{?}{=} a_\mu^{\mathrm{weak}} + a_\mu^{\mathrm{had}}$$

Weak contributions to a_μ: Z, W, H, etc. loops

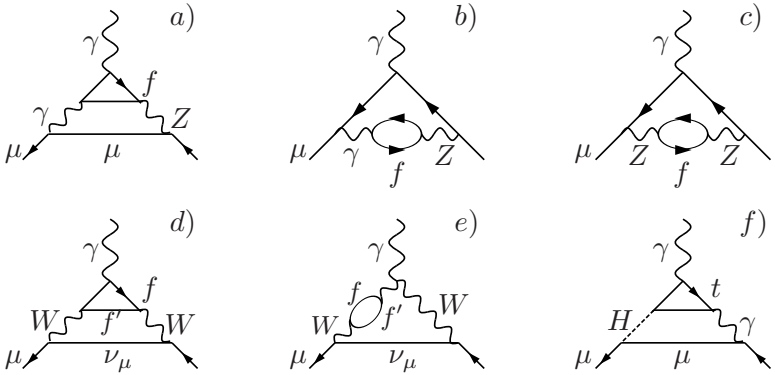
1-loop



$$a_\mu^{\text{weak},(1)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\right) = 19.480(1) \times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

2-loop



$$a_\mu^{\text{weak},(2)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\frac{\alpha}{\pi}\right) = -4.12(60) \times 10^{-10}$$

(Gnendiger et al '15 and refs therein)

$$a_\mu^{\text{weak}} = 15.36(10) \times 10^{-10}$$

Hadronic contributions to a_μ: quark and gluon loops

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{weak}} = 721.8(6.3) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{had}}$$

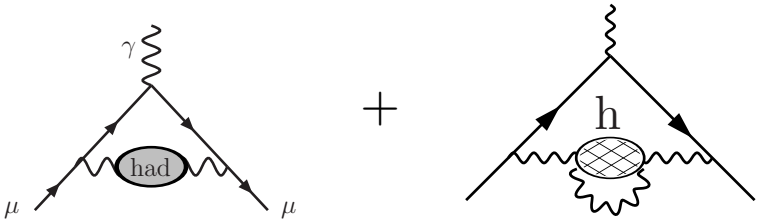
Clearly right order of magnitude:

$$a_\mu^{\text{had}} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

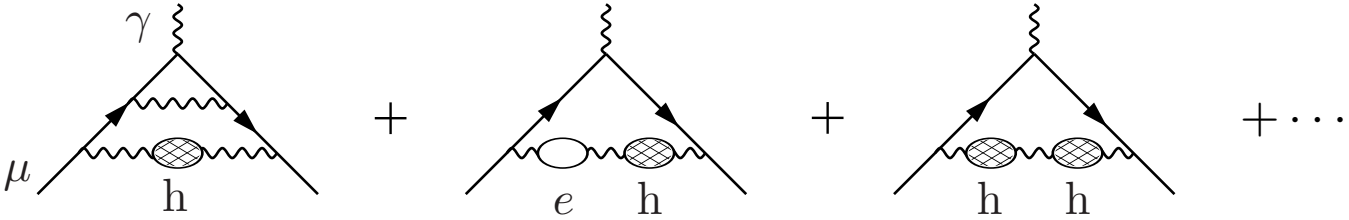
Write

$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{NLO-HVP}} + a_\mu^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$

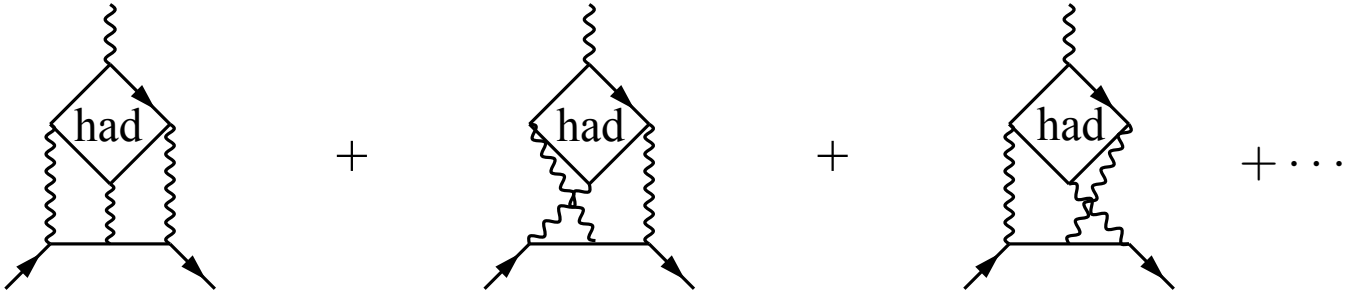
Hadronic contributions to a_μ: diagrams



$\rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$



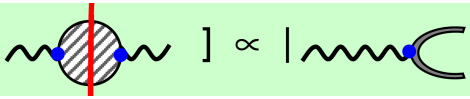
$\rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$



$\rightarrow a_\mu^{\text{HLbyL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

LO-HVP from $e^+e^- \to$ had

- Use (Bouchiat et al 61) optical theorem (unitarity)

Im[] $\propto$ | hadrons $|^2$ $\Rightarrow \mathrm{Im}\Pi(s) = -\frac{R(s)}{12\pi}, \qquad R(s) \equiv \frac{\sigma(e^+e^- \to \mathrm{had})}{4\pi\alpha(s)^2/(3s)}$

and a dispersion relation w/ data for $R(s)$ (CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.)

$$\hat{\Pi}(Q^2) = \int_0^\infty ds \, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s) = \frac{Q^2}{12\pi^2} \int_0^\infty ds \, \frac{1}{s(s+Q^2)} R(s)$$

- Three recent determinations:

$$\begin{aligned} a_\mu^{\text{LO-HVP}} &= 693.27(2.46) \times 10^{-10} \quad [3.5‰] \quad \text{(KNT '18)} \\ &= 693.1(3.4) \times 10^{-10} \quad [4.9‰] \quad \text{(DHMZ '17)} \\ &= 688.1(4.1) \times 10^{-10} \quad [6.0‰] \quad \text{(Jegerlehner '17)} \end{aligned}$$

- Higher orders:

$$\begin{aligned} a_\mu^{\text{NLO-HVP}} &= -9.87(0.09) \times 10^{-10} \quad \text{(Kurz et al '14)} \\ a_\mu^{\text{NNLO-HVP}} &= 1.24(0.01) \times 10^{-10} \quad \text{(Kurz et al '14)} \end{aligned}$$

Standard model prediction and comparison to experiment

SM prediction vs experiment

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{11}$	Ref.
QED [5 loops]	116584718.841 ± 0.034	[Aoyama et al '18]
HVP LO	6933 ± 25	[KNT '18]
	6931 ± 34	[DHMZ '17]
	6881 ± 41	[Jegerlehner '17]
HVP NLO	-98.7 ± 0.9	[Kurz et al '14]
		[Kurz et al '14, Jegerlehner '16]
HVP NNLO	12.4 ± 0.1	[Kurz et al '14]
		[Jegerlehner '16]
HLbyL	105 ± 26	[Prades et al '09]
	$54 \pm 14 \pm ??$	[RBC '16]
Weak (2 loops)	153.6 ± 1.0	[Gnendiger et al '15]
SM Tot [0.31 ppm]	116591824 ± 36	[w/ KNT '18]
[0.37 ppm]	116591822 ± 43	[w/ DHMZ '17]
[0.42 ppm]	116591772 ± 49	[Jegerlehner '17]
Exp [0.54 ppm]	116592091 ± 63	[Bennett et al '06]
Exp − SM	267 ± 72	[KNT '18]
	269 ± 76	[DHMZ '17]
	319 ± 80	[Jegerlehner '17]

a_μ present and future

- Today $a_\mu^{\rm exp} - a_\mu^{\rm SM} \simeq 3.5 \div 4.0\,\sigma$
- Fermilab E989 began fall 2017 and aims for 0.14 ppm
- J-PARC E34 to begin $\geq$ 2021 and aims for similar precision
- If both central values stay the same:
  - E989 alone (exp. error $/4$) $\rightarrow\sim 6 \div 7\,\sigma$
  - E989 + new HLbyL (w/ error $\sim 10\%$) $\rightarrow\sim 7 \div 9\,\sigma$
  - E989 + new HLbyL + new HVP (w/ error $/2$) $\rightarrow\sim 7 \div 12\,\sigma$
- Must have a completely independent first principles determination of the 2 contributions w/ leading theory errors before concluding the presence of BSM physics
  $\Rightarrow$ Lattice QCD
- Discrepancy is large: $2\times$ electroweak contribution
- No new physics scenario, i.e. (experiment) $-$ (other contributions)
  - $a_\mu^{\text{LO-HVP}}|_{\text{no-N}\Phi} = 720.0(6.8) \times 10^{-10}$, i.e. 4% larger than DHMZ '17
  - $a_\mu^{\text{HLbyL}}|_{\text{no-N}\Phi} = 37.9(7.1) \times 10^{-10}$, i.e. $3.6\times$ (Prades et al '09)

A brief introduction to lattice QCD

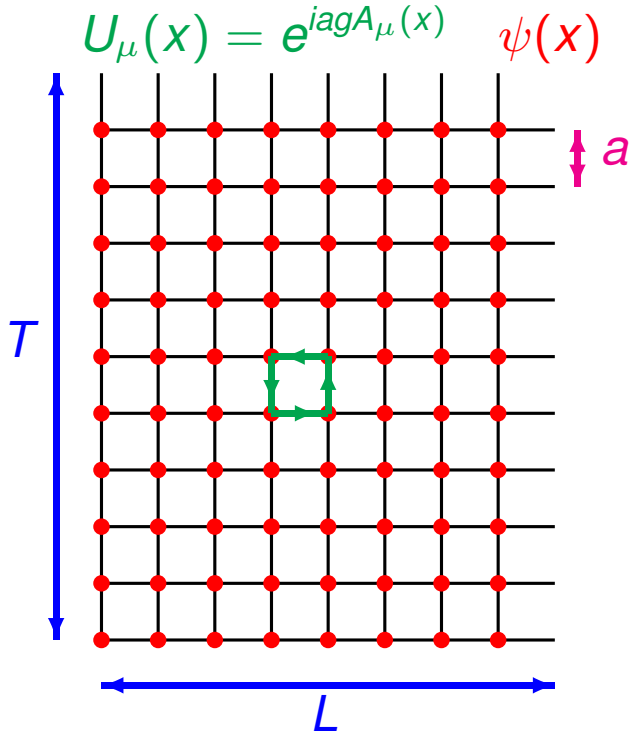
What is lattice QCD (LQCD)?

To describe ordinary matter, QCD requires ≥ 104 numbers at every point of spacetime
$\rightarrow \infty$ number of numbers in our continuous spacetime
$\rightarrow$ must temporarily "simplify" the theory to be able to calculate *(regularization)*
$\Rightarrow$ Lattice gauge theory $\longrightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$\begin{aligned}\langle O \rangle &= \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-S_G - \int \bar{\psi} D[M] \psi} \, O[U, \psi, \bar{\psi}] \\ &= \int \mathcal{D}U \, e^{-S_G} \det(D[M]) \, O[U]_{\text{Wick}}\end{aligned}$$

- $\mathcal{D}U e^{-S_G} \det(D[M]) \geq 0$ & finite # of dofs $\rightarrow$ evaluate numerically using stochastic methods



LQCD is QCD when $m_q \rightarrow m_q^{\text{ph}}$, $a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

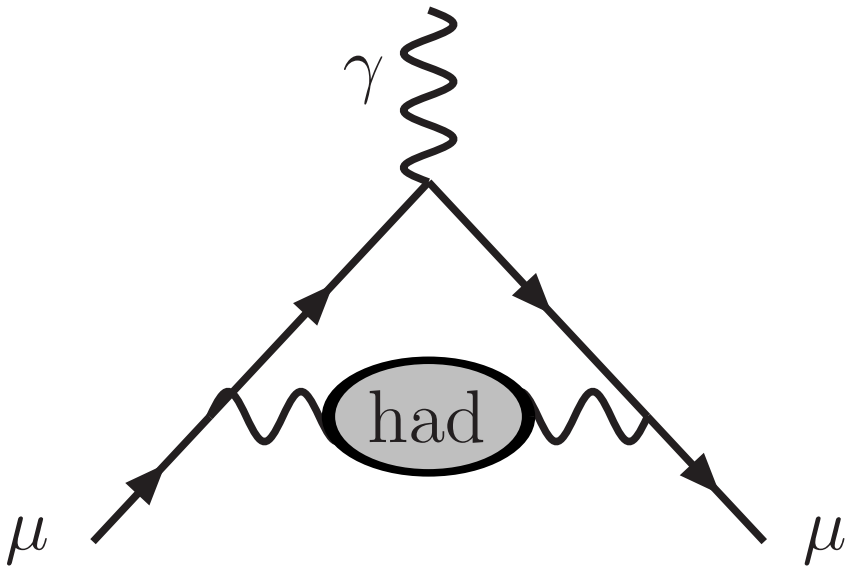
HUGE conceptual and numerical ($O(10^9)$ dofs) challenge

Challenges of a full lattice calculation

To make contact with experiment need:

- **A valid approximation to the SM**
  - $\rightarrow$ at least $u,\ d,\ s$ in the sea w/ $m_u = m_d \ll m_s$ (N_f=2+1)
  - $\rightarrow$ better also include c (N_f=2+1+1) & $m_u \leq m_d$ (N_f=4×1) & EM (N_f=4×1 + QED)
- **u & d w/ masses well w/in $SU(2)$ chiral regime** : $\sigma_\chi \sim (M_\pi/4\pi F_\pi)^2$
  - $\rightarrow$ $M_\pi \sim 135\,\mathrm{MeV}$ or many $M_\pi \leq 400\,\mathrm{MeV}$ w/ $M_\pi^{\mathrm{min}} < 200\,\mathrm{MeV}$ for $M_\pi \rightarrow 135\,\mathrm{MeV}$
- **a $\rightarrow$ 0** : $\sigma_a \sim (a\Lambda_{\mathrm{QCD}})^n,\ (am_q)^n,\ (a|\vec{p}|)^n$ w/ $a^{-1} \sim 2 \div 4\,\mathrm{fm}$
  - $\rightarrow$ at least 3 a's $\leq 0.1\,\mathrm{fm}$ for $a \rightarrow 0$
- **L $\rightarrow \infty$** : $\sigma_L \sim (M_\pi/4\pi F_\pi)^2 \times e^{-LM_\pi}$ for stable hadron pties, $\sim 1/L^n$ for resonances, QED, ...
  - $\rightarrow$ many L w/ $(LM_\pi)^{\mathrm{max}} \gtrsim 4$ for stable hadrons & better otherwise to allow for $L \rightarrow \infty$
- These requirements $\Rightarrow$ $O(10^9)$ **dofs** that have to be integrated over
- **Renormalization** : best done nonperturbatively
- **A signal** : $\sigma_{\mathrm{stat}} \sim 1/\sqrt{N_{\mathrm{meas}}}$, reduce w/ $N_{\mathrm{meas}} \rightarrow \infty$

Lattice QCD calculation of a_μ^{HVP}



HVP from LQCD: introduction

Consider in Euclidean spacetime (Blum '02)

$$\Pi_{\mu\nu}(Q) = \quad \gamma \; q \; q \; \gamma$$

$$= \int d^4x \, e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0) \rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2 \right) \Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

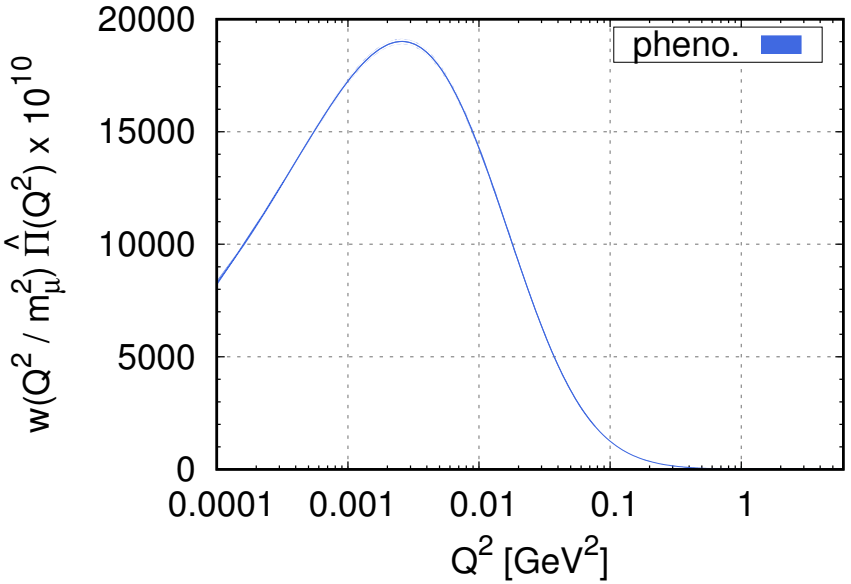
Then (Lautrup et al '69, Blum '02)

$$a_\ell^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dQ^2}{m_\ell^2} \, w(Q^2/m_\ell^2) \hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0) \right]$

Integrand peaked for $Q \sim (m_\ell/2) \sim 50\,\text{MeV}$ for μ

However, $Q_{\text{min}} \equiv \frac{2\pi}{T} \sim 135\,\text{MeV}$ for lattice w/ $T = \frac{3}{2}L \sim 9\,\text{fm}$



(HVP from Jegerlehner, "alphaQEDc17" (2017))

Low-Q^2 challenges in finite volume (FV)

A. Must subtract $\Pi_{\mu\nu}(Q=0) \neq 0$ in FV that contaminates $\Pi(Q^2) \sim \Pi_{\mu\nu}(Q)/Q^2$ for $Q^2 \to 0$ w/ very large FV effects

B. On-shell renormalization requires $\Pi(0)$ which is problematic (see above)

C. Need $\hat{\Pi}(Q^2)$ interpolation due to $Q_{\rm min} = 2\pi/T \sim 135\,\mathrm{MeV} > \frac{m_\mu}{2} \sim 50\,\mathrm{MeV}$ for $T \sim 9\,\mathrm{fm}$

$$\Downarrow$$

- Compute on $T \times L^3$ lattice

$$C_L(t) = \frac{a^3}{3} \sum_{i=1}^{3} \sum_{\vec{x}} \langle J_i(x) J_i(0) \rangle$$

- Decompose ($C_L^{I=1} = \frac{9}{10} C_L^{ud}$)

$$\begin{aligned} C_L(t) &= C_L^{ud}(t) + C_L^{s}(t) + C_L^{c}(t) + C_L^{\rm disc}(t) \\ &= C_L^{I=1}(t) + C_L^{I=0}(t) \end{aligned}$$

- Define (Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...) (ad A, B, C)

$$\hat{\Pi}_L^f(Q^2) \equiv \Pi_L^f(Q^2) - \Pi_L^f(0) = \frac{1}{3} \sum_{i=1}^{3} \frac{\Pi_{ii,L}^f(0) - \Pi_{ii,L}^f(Q)}{Q^2} - \Pi_L^f(0) = 2a \sum_{t=0}^{T/2} \mathrm{Re}\left[\frac{e^{iQt}-1}{Q^2} + \frac{t^2}{2} \right] \mathrm{Re} C_L^f(t)$$

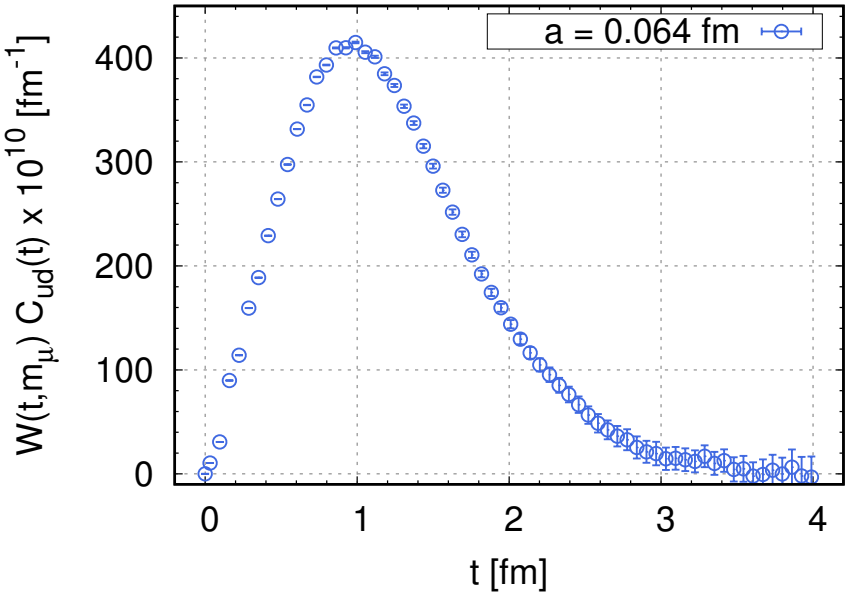
Our lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_L^f(t)$:

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \leq Q^2_{\max}) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m_\ell^2}\right) \sum_{t=0}^{T/2} W(tm_\ell, Q^2_{\max}/m_\ell^2)\ \mathrm{Re}C_L^f(t)$$

where

$$W(\tau, x_{\max}) = \int_0^{x_{\max}} dx\, w(x) \left(\tau^2 - \frac{4}{x}\sin^2\frac{\tau\sqrt{x}}{2}\right)$$

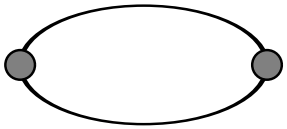


(144×96^3, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

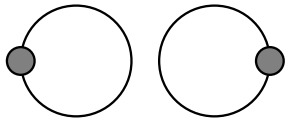
Simulation challenges

D. $\pi\pi$ contribution very important $\rightarrow$ have physically light π

E. Two types of contributions



quark-connected (qc)



quark-disconnected (qd)

where qd contributions are $SU(3)_f$ and Zweig suppressed but very challenging

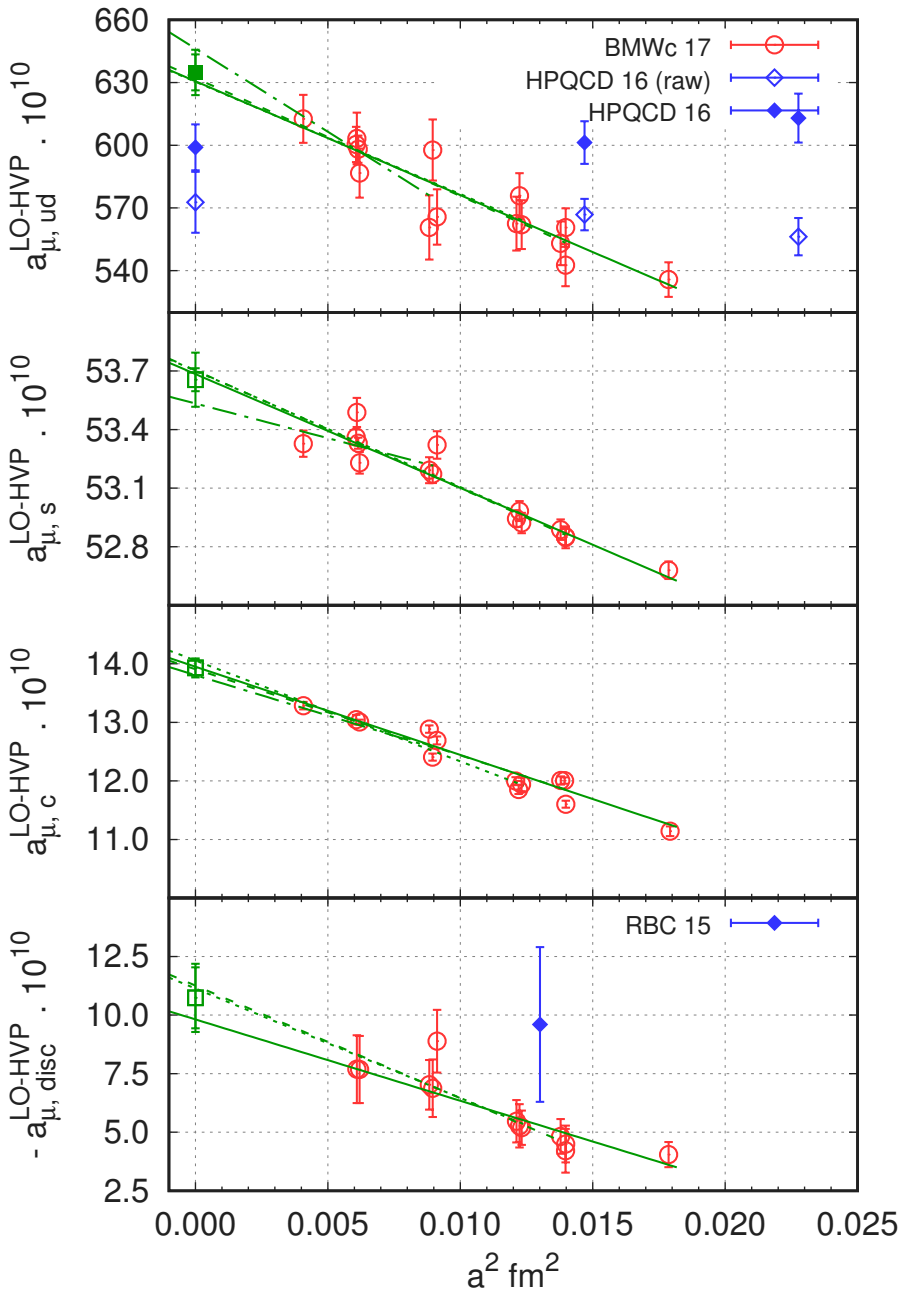
F. $\langle J_\mu^{ud}(x) J_\nu^{ud}(0)\rangle_{\rm qc}$ & disc. have very poor signal at large $\sqrt{x^2}$ + need high-precision results

$\rightarrow$ very high statistics + many algorithmic improvements + rigorous bounds
$\rightarrow$ 9M / 39M conn./disc. measurements

G. Must control $\langle J_\mu(x) J_\nu(0)\rangle$ at $\sqrt{x^2} \gtrsim 2/m_\mu$ $\rightarrow L = 6.1 \div 6.6\,\mathrm{fm}$, $T = 8.6 \div 11.3\,\mathrm{fm}$

H. Need controlled continuum limit $\rightarrow$ have 6 a's: $0.134 \rightarrow 0.064\,\mathrm{fm}$

Continuum limit of $a_{\mu,f}^{\text{LO-HVP}}(Q^2 \leq 4\,\text{GeV}^2)$



- With 6 a's, have full control over continuum limit
- Get good χ^2/dof w/ extrapolation linear in a^2 and interpolations, linear in M_π^2 and M_K^2
- Strong continuum extrapolation for $a_{\mu,ud/\text{disc}}^{\text{LO-HVP}}$ due to taste violations and for $a_{\mu,c}^{\text{LO-HVP}}$ due to large m_c
- Continuum extrapolation systematic from all results and by cutting results with $a \geq 0.134,\ 0.111,\ 0.095\,\text{fm}$
- Obtain other $a_{\mu,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}})$ and $\hat{\Pi}(Q_{\text{max}}^2)$, $Q_{\text{max}}^2 = 1, \cdots, 5\,\text{GeV}^2$, in entirely analogous fashion

More challenges

I. Need $\hat{\Pi}(Q^2)$ for $Q^2 \in [0, +\infty[$, but $\frac{\pi}{a} \sim 9.7\,\mathrm{GeV}$ for $a \sim 0.064\,\mathrm{fm}$
$\rightarrow$ match onto perturbation theory

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + \gamma_\ell(Q_{\text{max}})\,\hat{\Pi}^f(Q^2_{\text{max}}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}})$$

using $O(\alpha_s^4)$ results from Harlander et al '03

J. Include c quark for higher precision and good matching onto perturbation theory $\rightarrow$ done

K. Even in our large volumes w/ $L \gtrsim 6.1\,\mathrm{fm}$ & $T \geq 8.7\,\mathrm{fm}$, finite-volume (FV) effects can be significant

$\rightarrow$ correct using 1-loop SU(2) χPT (Aubin et al '16)

L. Our $N_f = 2 + 1 + 1$ calculation has $m_u = m_d$ and $\alpha = 0$

$\Rightarrow$ missing effects compared to HVP from dispersion relations that are relevant at %-level precision

$\rightarrow$ use phenomenology (F.Jegerlehner (& M. Benayoun), *private communication*)

Systematic errors and results for $a_\mu^{\text{LO-HVP}}$

- Stat. error: jackknife
- $a \to 0$: from 4 (3) cuts on a for conn. (disc.)
- bounds: from $t_c = 3.000(2.600) \pm 0.134\,\text{fm}$ vs $t_c = 2.866(2.466) \pm 0.134\,\text{fm}$ for conn. (disc.)
- PT match: from $Q^2_{\text{max}} = 2\,\text{GeV}^2$ vs $Q^2_{\text{max}} = 5\,\text{GeV}^2$
- $\delta a \simeq 0.4\% \Rightarrow \delta_a a_\mu^{\text{LO-HVP}} \simeq 0.8\%$
- FV: $a_{\mu,I=1}^{\text{LO-HVP}}(\infty) - a_{\mu,I=1}^{\text{LO-HVP}}(L{=}6\,\text{fm}) = (13.5 \pm 13.5) \times 10^{-10}$ from χPT
- IB: $\Delta_{\text{IB}} a_\mu^{\text{LO-HVP}} = (7.8 \pm 5.1) \times 10^{-10}$ from pheno.

Contrib.	$a_\mu^{\text{LO-HVP}} \times 10^{10}$
$I = 1$	583(7)(7)(0)(0)(5)(14)
$I = 0$	121(3)(4)(0)(0)(1)
Total	711(8)(8)(0)(0)(6)(13)(5)

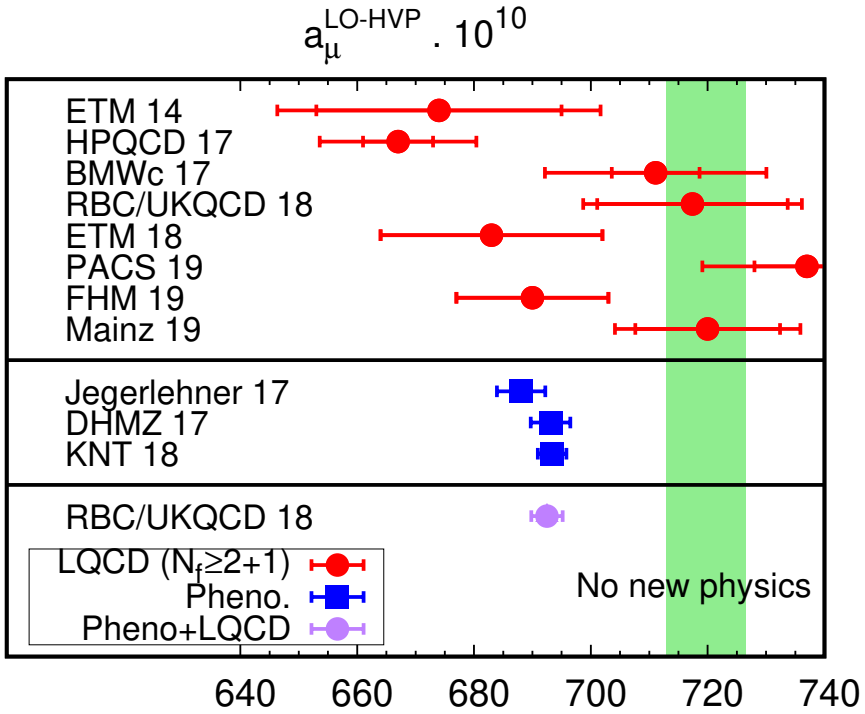
Error on total:

- Stat. = 1.1%
- LQCD syst. = 1.2%
- FV = 2.3%
- IB = 0.8%
- Total = 2.7%

Compare w/ upper bound (Bell et al '69) using Π_1 from BMWc, PRD96 $= 792(24)$

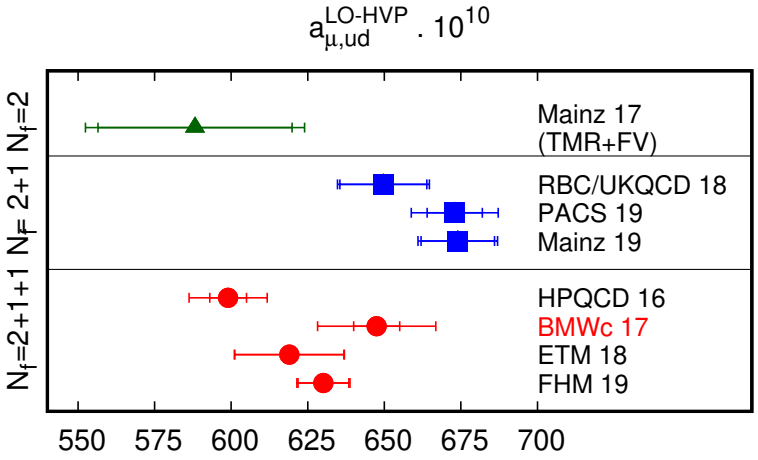
Comparison and outlook

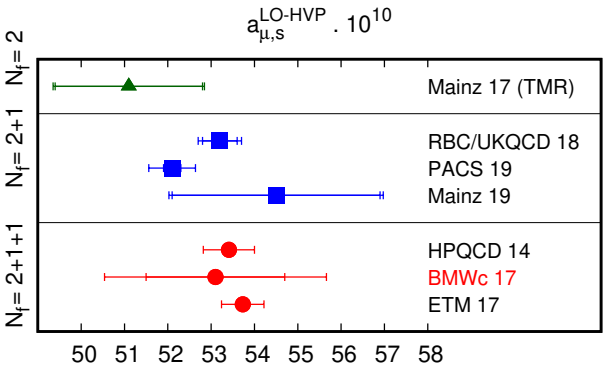
Comparison

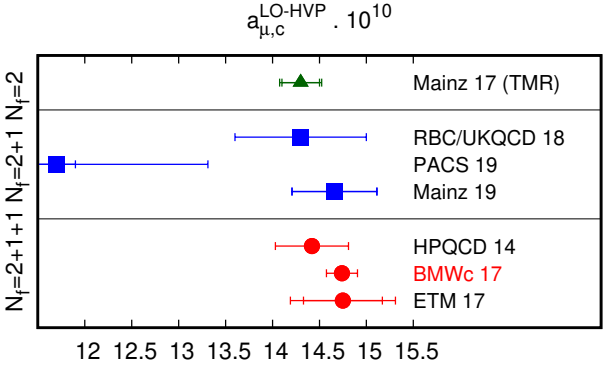


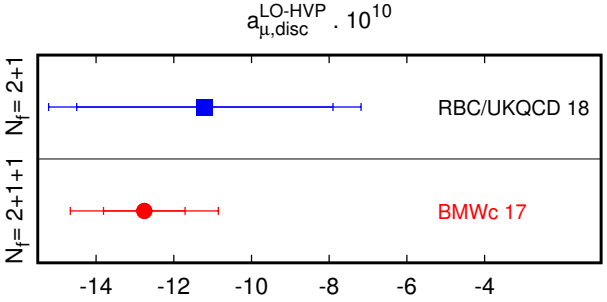
- "No New Physics" scenario: $= (720 \pm 7) \times 10^{-10}$
- BMWc '17 consistent w/ "No new physics" scenario & pheno.
- Total uncertainty of 2.7% is $\sim 6\times$ pheno. error
- BMWc '17 first result to indicate possibly larger result
  $\rightarrow$ since, all results compatible w/ that value

More detailed comparison









- all calculations are now compatible with BMWc '17 *ud* result
- BMWc '17 is only calculation performed directly at physical quark masses w/ 6 *a*'s to fully control continuum extrapolation
- most collaborations now agree on *s* and *c* contributions
- $a_{\mu,s,c,\text{disc}}^{\text{LO-HVP}}$ already known with high enough precision for FNAL E989

Where we are

- Calculation of all relevant contributions to $a_\mu^{\text{LO-HVP}}$ directly at physical m_{ud}

$$a_\mu^{\text{LO-HVP}} = 711.0(7.5)(17.5) \times 10^{-10} \quad [2.7\%]$$

- Also have slope and curvature of $\hat{\Pi}(Q^2)$ at $Q^2 = 0$ (PRD96 '17)
- Fully controlled continuum limit and matching to perturbation theory
- Only model/pheno. assumptions for FV, QED and $m_u \neq m_d$ corrections
- Consistent with "no new physics" and dispersive methods, but error $\sim 6\times$ larger; some tension with HPQCD 16 on $a_{\mu,ud}^{\text{LO-HVP}}$, partially resolved in FHM
- Total error is 2.7%, dominated by poorly controlled FV effects
- Need $\sim 0.2\%$ to match upcoming experiments !
- With same methods, compute (see also ETM '16)

$$a_e^{\text{LO-HVP}} = 189.3(2.6)(5.6) \times 10^{-14}[3.2\%] \leftrightarrow 184.6(1.2) \times 10^{-14}[0.7\%] \quad \text{(Jegerlehner '15)}$$

$$a_\tau^{\text{LO-HVP}} = 341.3(0.8)(3.2) \times 10^{-8}[1.0\%] \leftrightarrow 338(4) \times 10^{-8}[1.2\%] \quad \text{(Eidelman et al. '07)}$$

What next?

- Need to reduce our error by 10!

→ Increase statistics by $\times 50 \div 100$ (need new methods)

→ Understand and control FV effects much better

→ Compute QED and $m_d \neq m_u$ corrections (see RBC/UKQCD '17-'18, ETM '17)

→ Need high precision scale setting

→ Detailed comparison to phenomenology to understand where we agree and why if we don't

→ Combine LQCD and phenomenology to improve overall uncertainty (RBC/UKQCD '18), only if the two agree statistically with comparable errors