

On generally covariant mathematical formulation of Feynman integral in Lorentz signature

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Outline

- Structure of model building in fundamental physics.
- Model building attempts in QFT.
- Heuristic Feynman integral formulation.
- Classical field theory.
- Rigorous form of master Dyson-Schwinger (MDS) equation.
- Existence condition for regularized MDS solutions.
- Outlook: tentative rigorous definition for the Wilsonian renormalization.
- Summary.

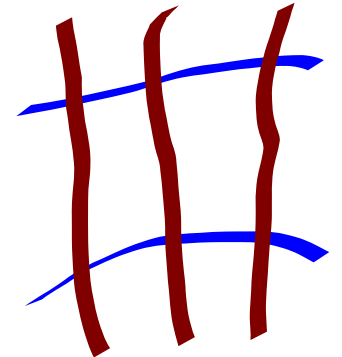
Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Space of solutions turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space.

Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Space of finite charge weak solutions turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space.



Most important ingredient: a one-liner, the equation of motion or field equation.

→ Then, one is working on the solution space.

Can one find a one-liner equation to summarize Quantum Field Theory?

Model building attempts in QFT

Common QFT formalisms in physics:

- Often non-manifestly covariant formalism.
(Hamiltonian, reminescent of non-relativistic QM as seen by an inertial observer.)
- In momentum space.
- Splitting Lagrangian to free + interacting terms.
- Often perturbative handling.
- Need for regularization and renormalization. (What this is precisely?)
- Not easy to see what is legitimate and what is not.
- In some cases the “right” thing is done, even without the adequate formalism.

Common formalisms in mathematical QFT:

- Loop quantum gravity. (Spacetime is emergent, far from finalized.)
- Algebraic QFT: easy to understand math/physics concept, no known 3+1d example.
- Perturbative AQFT formalized over fixed spacetime, known examples.
(Still cannot put down a one-liner.)
- Feynman integral in Wick rotated signature. But still free + interaction splitting.

Our guidelines:

- Do not assume spacetime manifold to be affine space. (Spacetime not flat.)
- Do not refer to fixed spacetime metric / causal structure. (Do not exclude dynamical c.s.)
- Only refer to underlying spacetime manifold and generic fields over it.
(Metric not distinguished field – but spacetime manifold usual 4d continuum.)
- Do not refer to known splitting of Lagrangian to free + interaction parts.
(No canonical way to split e.g. Yang-Mills or Einstein-Hilbert Lagrangian.)

Some consequences:

- Cannot go to momentum space, have to stay in spacetime description.
- Cannot refer to any affine property of Minkowski spacetime, e.g. asymptotics.
(No notion of space \mathcal{S} of rapidly decreasing functions.)
- Cannot use Wick rotation to Euclidean signature metric.
- Even if Wick rotated, no free + interaction splitting, so no Gaussian Feynman measure.

We revisit Feynman integral formulation in Lorentz signature.

[We conclude that its differential reformulation can be well defined.]

Heuristic Feynman integral formulation

Let \mathcal{M} be smooth manifold (wannabe spacetime, but no metric, yet).

Let $V(\mathcal{M})$ be vector bundle over it (its smooth sections are matter fields – also metric here).

On this, there are covariant derivation ops (gauge fields), they form affine bundle $DV(\mathcal{M})$.

Affine bundle over YM potentials, i.e. sections of $T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M}) =: CV(\mathcal{M})$.

In total:

$$\underbrace{(v, \nabla)}_{=: \psi} \in \underbrace{\Gamma(V(\mathcal{M}) \times_{\mathcal{M}} DV(\mathcal{M}))}_{=: F} \text{ is a field configuration.}$$

These with the \mathcal{E} smooth function topology form a real topological affine space.

$$\underbrace{(\delta v, \delta C)}_{=: \delta\psi} \in \underbrace{\Gamma(V(\mathcal{M}) \times_{\mathcal{M}} CV(\mathcal{M}))}_{=: \mathbb{F}} \text{ is a field variation, difference of two field configs.}$$

These with the \mathcal{E} smooth function topology form a real topological vector space.

Fix some $\psi_0 \in F$ reference field for bringing the problem to field variations ($F \rightarrow \mathbb{F}$).

Let $J_1, \dots, J_n \in \mathbb{F}^*$ be test functionals.

Then, Feynman type quantum expectation value of polynomial observable

$(J_1 | \cdot - \psi_0) \cdot \dots \cdot (J_n | \cdot - \psi_0) : F \rightarrow \mathbb{R}$ in vacuum state ρ postulated as:

$$\int_{\psi \in F} (J_1 | \psi - \psi_0) \cdot \dots \cdot (J_n | \psi - \psi_0) e^{\frac{i}{\hbar} S(\psi)} [d\psi]_\rho \bigg/ \int_{\psi \in F} e^{\frac{i}{\hbar} S(\psi)} [d\psi]_\rho$$

Partition function is often invoked to book-keep all this (Fourier transform of $e^{iS(\psi)} [d\psi]_\rho$):

$$Z_{\psi_0, \rho} : \mathbb{F}^* \longrightarrow \mathbb{C}, \quad J \longmapsto Z_{\psi_0, \rho}(J) := \int_{\psi \in F} e^{i(J | \psi - \psi_0)} e^{\frac{i}{\hbar} S(\psi)} [d\psi]_\rho,$$

and from this one can define

$$G_{\psi_0, \rho}^{(n)} := \left((-i)^n \frac{1}{Z_{\psi_0, \rho}(J)} D^{(n)} Z_{\psi_0, \rho}(J) \right) \bigg|_{J=0}$$

n -field correlator, and their collection $G_{\psi_0, \rho} := \left(G_{\psi_0, \rho}^{(0)}, G_{\psi_0, \rho}^{(1)}, \dots, G_{\psi_0, \rho}^{(n)}, \dots \right) \in \bigoplus_{n \in \mathbb{N}_0}^n \mathbb{F}$.

Above Feynman type quantum expectation value expressible as:

$$\left(J_1 \otimes \dots \otimes J_n \mid G_{\psi_0, \rho}^{(n)} \right)$$

Problem: no “Lebesgue” measure $[d\psi]_\rho$ in infinite dimensions.

Neither $e^{\frac{i}{\hbar} S(\psi)} [d\psi]_\rho$ is meaningful. (Can be given some meaning in Euclidean signature.)

Neither formal Fourier transformation of this undefined measure is meaningful.

In usual QFT literature, $e^{\frac{i}{\hbar} S(\psi)} [d\psi]_\rho$ is handled as if it existed as **finite measure**, with **finite moments** and **analytic Fourier transform**.

Formally playing with Fourier transform, one infers:

$Z : \mathbb{F}^* \rightarrow \mathbb{C}$ Fourier transform of $e^{\frac{i}{\hbar} S(\psi)} [d\psi]_\rho$ “ \Leftrightarrow ” satisfies master-Dyson-Schwinger eq:

$$\left(\mathbf{E}((-i)D_J + \psi_0) + \hbar J \right) Z(J) = 0 \quad (\forall J \in \mathbb{F}^*)$$

with $E(\psi) := DS(\psi)$ being the Euler-Lagrange functional at $\psi \in F$.

Looks kind of weird to interpret beyond formality. Does it have a meaning?

Yes, when expressed via field correlators $G = (G^{(0)}, G^{(1)}, \dots, G^{(n)}, \dots)$.

Classical field theory

Let a **Lagrange form** be given, which is

$$L : V(\mathcal{M}) \oplus T^*(\mathcal{M}) \otimes V(\mathcal{M}) \oplus T^*(\mathcal{M}) \wedge T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M}) \longrightarrow \bigwedge^{\dim(\mathcal{M})} T^*(\mathcal{M})$$

pointwise vector bundle homomorphism.

Lagrangian expression:

$$\Gamma(V(\mathcal{M}) \times_{\mathcal{M}} DV(\mathcal{M})) \longrightarrow \Gamma\left(\bigwedge^{\dim(\mathcal{M})} T^*(\mathcal{M})\right), \quad (v, \nabla) \longmapsto L(v, \nabla v, P(\nabla))$$

where $P(\nabla)$ is the curvature tensor.

Action functional:

$$S : \underbrace{\Gamma(V(\mathcal{M}) \times_{\mathcal{M}} DV(\mathcal{M}))}_{=: F} \longrightarrow \text{Meas}(\mathcal{M}, \mathbb{R}), \quad \underbrace{(v, \nabla)}_{=: \psi} \longmapsto \left(\mathcal{K} \mapsto S_{\mathcal{K}}(v, \nabla) \right)$$

where $S_{\mathcal{K}}(v, \nabla) := \int_{\mathcal{K}} L(v, \nabla v, P(\nabla))$ for all $\mathcal{K} \subset \mathcal{M}$ compact.

[Achtung: $S_{\mathcal{M}}(v, \nabla)$ generally not finite, e.g. already for stationary fields etc.]

Action functional $S : F \rightarrow \text{Meas}(\mathcal{M}, \mathbb{R})$ Fréchet differentiable, its Fréchet derivative

$$DS : F \times \mathbb{F} \longrightarrow \text{Meas}(\mathcal{M}, \mathbb{R}), \quad (\psi, \delta\psi) \longmapsto \left(\mathcal{K} \mapsto (DS_{\mathcal{K}}(\psi) \mid \delta\psi) \right)$$

is the usual Euler-Lagrange integral on \mathcal{K} + usual boundary integral on $\partial\mathcal{K}$.

Jointly continuous in its variables, linear in second variable.

Let \mathbb{F}_T be the compactly supported field variations in \mathbb{F} with usual \mathcal{D} test function topology.
(space of test field variations)

Euler-Lagrange functional:

We restrict DS in its second variable to \mathbb{F}_T , to make the EL integral over full \mathcal{M} finite.

$$E : F \times \mathbb{F}_T \longrightarrow \mathbb{R}, \quad (\psi, \delta\psi_T) \longmapsto (E(\psi) \mid \delta\psi_T) := (DS_{\mathcal{M}}(\psi) \mid \delta\psi_T)$$

Bulk Euler-Lagrange integral remains, no boundary term. Meaningful on full \mathcal{M} , real valued.

Jointly sequentially continuous, linear in second variable. (Also, $E : F \rightarrow \mathbb{F}_T^*$ continuous.)

Classical field equation is

$$\psi \in F ? \quad \forall \delta\psi_T \in \mathbb{F}_T : (E(\psi) \mid \delta\psi_T) = 0.$$

Observables are the $O : F \rightarrow \mathbb{R}$ continuous maps.

Example: φ^4 theory.

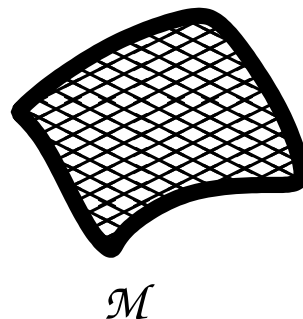
\mathcal{M} is Minkowski spacetime, v is volume measure, \square is wave operator.

$F := \mathbb{F} := C^\infty(\mathcal{M}, \mathbb{R})$ and $\mathbb{F}_T := C_c^\infty(\mathcal{M}, \mathbb{R})$.

Euler-Lagrange functional is

$$E : F \times \mathbb{F}_T \longrightarrow \mathbb{R}, \quad (\psi, \delta\psi_T) \longmapsto \int_{\mathcal{M}} \delta\psi_T \square\psi \, v + \int_{\mathcal{M}} \delta\psi_T \psi^3 \, v.$$

Field equation selects physically realizable fields over spacetime manifold.



Rigorous form of the master Dyson-Schwinger equation

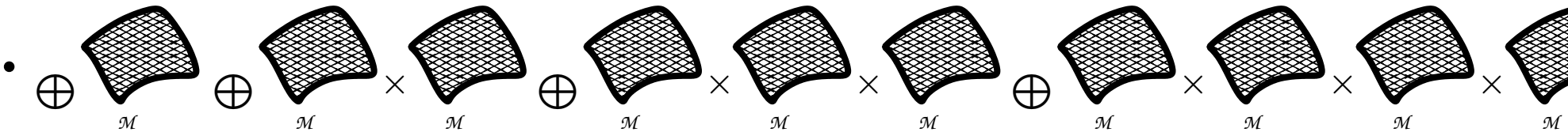
We want an equation determining the physically realizable n -field correlators ($n = 0, 1, 2, \dots$).

Candidates sit in the tensor algebra $\mathcal{T}(\mathbb{F}) := \bigoplus_{n \in \mathbb{N}_0} \bigotimes^n \mathbb{F}$ of field variations.

More precisely, they sit in a graded-symmetrized subspace, e.g. $\bigvee(\mathbb{F})$ or $\bigwedge(\mathbb{F})$ of $\mathcal{T}(\mathbb{F})$.

They are tuples $G = (G^{(0)}, G^{(1)}, G^{(2)}, \dots, G^{(n)}, \dots)$ of n -variate classical fields.

(Over n copies of spacetime manifold, $n = 0, 1, 2, \dots$)



The multivariate fields $\bigotimes^n \mathbb{F}$ inherit a natural \mathcal{E} smooth function topology from \mathbb{F} .
(Schwartz kernel theorem.)

The tensor algebra $\mathcal{T}(\mathbb{F})$ inherits a natural (Tychonoff) topology from each $\bigotimes^n \mathbb{F}$, entrywise.
(Theorem: similar in nature to \mathcal{E} smooth function topology.)

It is a theorem that $(\mathcal{T}(\mathbb{F}))^* \equiv \mathcal{T}_a(\mathbb{F}^*)$ and that $(\mathcal{T}(\mathbb{F}))^{**} \equiv \mathcal{T}(\mathbb{F})$.

By construction, unital associative algebra, unit element is $\mathbb{1} := (1, 0, 0, 0, \dots)$

It is topological algebra, so left-multiplication $L_{\delta\psi}$ by a $\delta\psi \in \mathbb{F}$ meaningful and continuous.

Theorem:

left-insertion ι_p (tracing out) by some $p \in (\mathcal{T}(\mathbb{F}))^* \equiv \mathcal{T}_a(\mathbb{F}^*)$ also meaningful, continuous.

As usual

$$\left(\iota_p L_{\delta\psi} \pm L_{\delta\psi} \iota_p \right) G = (p|\delta\psi) G \quad (\forall p \in \mathbb{F}^* \text{ and } \delta\psi \in \mathbb{F} \text{ and } G)$$

graded-commutation relation.

[Important properties behave analogously as if \mathbb{F} were finite dimensional.]

Take a classical observable $O : F \rightarrow \mathbb{R}$, $\psi \mapsto O(\psi)$, let $O_{\psi_0} := O \circ (\mathbf{I}_F + \psi_0)$.

[One has $O_{\psi_0}(\psi - \psi_0) = O(\psi) \quad \forall \psi \in F$, with some fixed reference field $\psi_0 \in F$.]

We say that O is **multipolynomial** iff for some $\psi_0 \in F$ there exists $\mathbf{O}_{\psi_0} \in \mathcal{T}_a(\mathbb{F}^*)$, such that

$$\forall \psi \in F : \quad \underbrace{O_{\psi_0}(\psi - \psi_0)}_{= O(\psi)} = \left(\mathbf{O}_{\psi_0} \mid (1, \overset{1}{\otimes}(\psi - \psi_0), \overset{2}{\otimes}(\psi - \psi_0), \dots) \right).$$

Similarly $E : F \rightarrow \mathbb{F}_T^*$, $\psi \mapsto E(\psi)$, let $E_{\psi_0} := E \circ (I_{\mathbb{F}} + \psi_0)$ the same re-expressed on \mathbb{F} .

[One has $E_{\psi_0}(\psi - \psi_0) = E(\psi) \quad \forall \psi \in F$, with some fixed reference field $\psi_0 \in F$.]

We say that E is **multipolynomial** iff $\exists \mathbf{E}_{\psi_0} \in \mathcal{T}_a(\mathbb{F}^*) \otimes \mathbb{F}_T^*$, such that

$$\forall \psi \in F, \delta\psi_T \in \mathbb{F}_T : \underbrace{\left(E_{\psi_0}(\psi - \psi_0) \mid \delta\psi_T \right)}_{= (E(\psi) \mid \delta\psi_T)} = \left(\mathbf{E}_{\psi_0} \mid (1, \overset{1}{\otimes}(\psi - \psi_0), \overset{2}{\otimes}(\psi - \psi_0), \dots) \otimes \delta\psi_T \right).$$

For fixed $\delta\psi_T \in \mathbb{F}_T$, one has $(\mathbf{E}_{\psi_0} \mid \delta\psi_T) \in \mathcal{T}_a(\mathbb{F}^*)$.

So one can left-insert with it on the field correlator algebra:

$\mathcal{L}_{(\mathbf{E}_{\psi_0} \mid \delta\psi_T)}$ meaningfully acting on $\mathcal{T}(\mathbb{F})$.

The master Dyson-Schwinger (MDS) equation is:

we search for (ψ_0, G_{ψ_0}) such that:

$$\underbrace{G_{\psi_0}^{(0)}}_{=: b G_{\psi_0}} = 1,$$

$$\forall \delta\psi_T \in \mathbb{F}_T : \quad \underbrace{\left(\mathcal{L}_{(\mathbf{E}_{\psi_0} \mid \delta\psi_T)} - i \hbar L_{\delta\psi_T} \right)}_{=: \mathbf{M}_{\psi_0, \delta\psi_T}} G_{\psi_0} = 0.$$

We argue that this is the tentative “one-liner” of QFT.

It says \sim spontaneous local excitations decay in all modes according to classical EL.

[Feynman type quantum expectation value is then $\mu_{(\psi_0, G_{\psi_0})}(O) := (\mathbf{O}_{\psi_0} \mid G_{\psi_0}).$]

Example: ϕ^4 model.

Euler-Lagrange functional is

$$E : F \times \mathbb{F}_T \longrightarrow \mathbb{R}, \quad (\psi, \delta\psi_T) \longmapsto \int_{\mathcal{M}} \delta\psi_T \square\psi \, v + \int_{\mathcal{M}} \delta\psi_T \psi^3 \, v.$$

\Downarrow

MDS operator at $\psi_0 = 0$ reads

$$(\mathbf{M}_{\psi_0, \delta\psi_T} G)^{(n)}(x_1, \dots, x_n) =$$

$$\int_{y \in \mathcal{M}} \delta\psi_T(y) \square_y G^{(n+1)}(y, x_1, \dots, x_n) v(y) + \int_{y \in \mathcal{M}} \delta\psi_T(y) G^{(n+3)}(y, y, y, x_1, \dots, x_n) v(y)$$

$$-i \hbar \underbrace{n \frac{1}{n!} \sum_{\pi \in \Pi_n} \delta\psi_T(x_{\pi(1)}) G^{(n-1)}(x_{\pi(2)}, \dots, x_{\pi(n)})}_{(L_{\delta\psi_T} G)^{(n)}(x_1, \dots, x_n)}$$

Pretty much well-defined, and clear recipe, if field correlators were functions.

Theorem: no solutions with high differentiability (e.g. as smooth functions).

Theorem: for free Minkowski KG case, distributional solution only.

Dang!

Distributional solution to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$\begin{aligned} K_{\psi_0}^{(0)} &= 0, \\ K_{\psi_0}^{(1)} &= 0, \\ K_{\psi_0}^{(2)} &= i \hbar K_{\psi_0}^{(2)} \quad \longleftarrow \text{(symmetric propagator)} \\ K_{\psi_0}^{(n)} &= 0 \quad (n \geq 2) \end{aligned}$$

So we expect distributional solutions only, at best.

How can one evaluate on distributions interaction term like $G^{(n+3)}(y, y, y, x_1, \dots, x_n)$?

With a sufficient condition called Hörmander's criterion? (Theorem: no.)

Via approximation with functions? (Theorem: no.)

Surprising solution by physicists: [Wilsonian regularization](#).

Feynman integral “ \Longleftrightarrow ” MDS equation.

Wilsonian regularized Feynman integral:

integrate not on \mathbb{F} , only on the image space $C_\kappa[\mathbb{F}]$ of a smoothing operator $C_\kappa : \mathbb{F} \rightarrow \mathbb{F}$.

[Smoothing operator: \sim convolution, can be generalized to manifolds. Does UV damping.]

Wilsonian regularized Feynman integral “ \Longleftrightarrow ” regularized MDS equation:

we search for $(\psi_0, G_{\psi_0, \kappa})$ such that:

$$\underbrace{G_{\psi_0, \kappa}^{(0)}}_{=: b G_{\psi_0, \kappa}} = 1,$$

$$\forall \delta\psi_T \in \mathbb{F}_T : \quad \underbrace{\left(\ell_{(\mathbf{E}_{\psi_0} | \delta\psi_T)} - i \hbar L_{C_\kappa \delta\psi_T} \right)}_{=: \mathbf{M}_{\psi_0, \kappa, \delta\psi_T}} G_{\psi_0, \kappa} = 0.$$

Removes obstruction theorem for solutions with high differentiability.

Thus, brings back the problem from distributions to smooth functions.

Smooth function solution to free regularized MDS equation: $G_{\psi_0, \kappa} = \exp(K_{\psi_0, \kappa})$ where

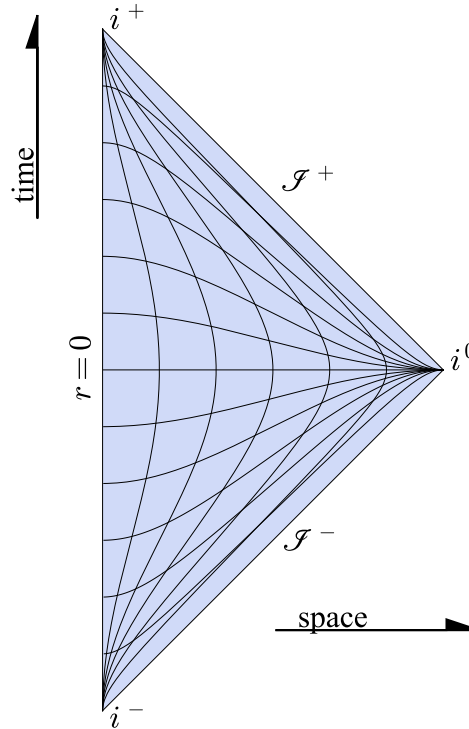
$$\begin{aligned} K_{\psi_0, \kappa}^{(0)} &= 0, \\ K_{\psi_0, \kappa}^{(1)} &= 0, \\ K_{\psi_0, \kappa}^{(2)} &= i \hbar K_{\psi_0, \kappa}^{(2)} \quad \longleftarrow \text{(smoothed symmetric propagator)} \\ K_{\psi_0, \kappa}^{(n)} &= 0 \quad (n \geq 2) \end{aligned}$$

No problem to evaluate on interaction term like $G^{(n+3)}(y, y, y, x_1, \dots, x_n)$.

What we do with κ dependence? (Renormalization, work in progress with Zs.Tarcsay.)

Existence condition for regularized MDS solutions

If Euler-Lagrange functional $E : F \rightarrow \mathbb{F}_T^*$ conformally invariant:
re-expressable on Penrose conformal compactification.



That is always a compact manifold with boundary, with sufficiently regular boundary.

$E : F \rightarrow \mathbb{F}_T^*$ reformulable over this compact base manifold with regular boundary.

So, one can assume \mathcal{M} compact with nice enough boundary.

In such situation, $\mathbb{F} = \mathbb{F}_T$ and have nice properties:
countably Hilbertian nuclear Fréchet space.

$$F_0 \supset F_1 \supset \dots \supset F_m \supset \dots \supset \mathbb{F}$$

(Intersection of shrinking Hilbert spaces F_m with Hilbert-Schmidt embedding.)

Theorem:

without penalty, one can equip $\mathcal{T}(\mathbb{F})$ with a better topology, inheriting CHNF topology.

$$H_0 \supset H_1 \supset \dots \supset H_m \supset \dots \supset \mathcal{T}_h(\mathbb{F})$$

Regularized MDS operator is then a Hilbert-Schmidt linear map

$$\mathbf{M}_{\psi_0, \kappa} : H_m \otimes F_m \longrightarrow H_0, \quad G \otimes \delta\psi_T \longmapsto \mathbf{M}_{\psi_0, \kappa, \delta\psi_T} G$$

Theorem: one can legitimately trace out $\delta\psi_T$ variable to form

$$\hat{\mathbf{M}}_{\psi_0, \kappa}^2 : H_m \longrightarrow H_m, \quad G \longmapsto \sum_{i \in \mathbb{N}_0} \mathbf{M}_{\psi_0, \kappa, \delta\psi_T i}^\dagger \mathbf{M}_{\psi_0, \kappa, \delta\psi_T i} G$$

By construction: G is κ -regularized MDS solution $\iff bG = 1$ and $\hat{\mathbf{M}}_{\psi_0, \kappa}^2 G = 0$.

Theorem (A.László):

(i) the iteration

$$G_0 := \mathbb{1} \text{ and } G_{l+1} := G_l - \frac{1}{T} \hat{\mathbf{M}}_{\psi_0, \kappa}^2 G_l \quad (l = 0, 1, 2, \dots)$$

is always convergent if $T > 0$ large enough.

(ii) the κ -regularized MDS solution space is nonempty iff

$$\lim_{l \rightarrow \infty} bG_l \neq 0.$$

(iii) and in this case

$$\lim_{l \rightarrow \infty} G_l$$

is an MDS solution, up to normalization factor.

Use for lattice-like numerical method in Lorentz signature?

Summary

- Feynman integral has no rigorous definition in Lorentz signature.
- Can be substituted by master Dyson-Schwinger (MDS) equation.
- Function spaces and operators for MDS equation are well defined (in suitable variables).
- Wilsonian regularized version of MDS equation is well defined (in suitable variables).
- Does not need a pre-arranged fixed causal structure.
- Necessary and sufficient existence condition proved for MDS solutions.
- Provides a convergent iterative approximation algorithm.
(Lattice in Lorentz signature?)
- Outlook: rigorous definition of Wilsonian renormalization.
(Joint work in progress with Zsigmond Tarcsay.)

Backup slides

Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Solution space to the equation turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space:
 - Elements of solution space X are elementary events.
 - Collection of Borel sets Σ of X are composite events.
 - A state is a probability measure W on Σ , i.e. (X, Σ, W) is classical probability space.

Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Finite charge weak solution space to the equation turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space:
 - One dimensional subspaces of the solution space \mathcal{H} are elementary events, X .
 - Collection of all closed subspaces Σ of \mathcal{H} are composite events.
 - A state is a probability measure W on Σ , i.e. (X, Σ, W) is quantum probability space.

Fréchet derivative in top.vector spaces

Let F and G real top.affine space, Hausdorff.

Subordinate vector spaces: \mathbb{F} and \mathbb{G} .

A map $S : F \rightarrow G$ is **Fréchet-Hadamard differentiable at $\psi \in F$** iff:

there exists $DS(\psi) : \mathbb{F} \rightarrow \mathbb{G}$ continuous linear, such that for all sequence $n \mapsto h_n$ in \mathbb{F} , and nonzero sequence $n \mapsto t_n$ in \mathbb{R} which converges to zero,

$$(\mathbb{G}) \lim_{n \rightarrow \infty} \left(\frac{S(\psi + t_n h_n) - S(\psi)}{t_n} - DS(\psi) h_n \right) = 0$$

holds.

Fréchet derivative of action functional

Fréchet derivative of $S : F \longrightarrow \text{Meas}(\mathcal{M}, \mathbb{R})$ is

$$DS : F \times \mathbb{F} \longrightarrow \text{Meas}(\mathcal{M}, \mathbb{R}), (\psi, \delta\psi) \longmapsto \left(\mathcal{K} \mapsto (DS_{\mathcal{K}}(\psi) | \delta\psi) \right)$$

For $\underbrace{(v, \nabla)}_{=: \psi} \in F$ given,

$$\underbrace{(\delta v, \delta C)}_{=: \delta\psi} \mapsto (DS_{\mathcal{K}}(v, \nabla) | (\delta v, \delta C)) =$$

$$\begin{aligned} & \int_{\mathcal{K}} \left(D_1 L(v, \nabla v, P(\nabla)) \delta v + D_2^a L(v, \nabla v, P(\nabla)) (\nabla_a \delta v + \delta C_a v) + 2 D_3^{[ab]} L(v, \nabla v, P(\nabla)) \tilde{\nabla}_{[a} \delta C_{b]} \right) \\ &= \int_{\mathcal{K}} \left(D_1 L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta v - (\tilde{\nabla}_a D_2^a L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta v \right) + \\ & \quad \left(D_2^a L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta C_a v - 2 (\tilde{\nabla}_a D_3^{[ab]} L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta C_b \right) \\ &+ m \int_{\partial \mathcal{K}} \left(D_2^a L(v, \nabla v, P(\nabla))_{[a c_1 \dots c_{m-1}]} \delta v + 2 D_3^{[ab]} L(v, \nabla v, P(\nabla))_{[a c_1 \dots c_{m-1}]} \delta C_b \right) \end{aligned}$$

$$(m := \dim(\mathcal{M}))$$

[usual Euler-Lagrange bulk integral + boundary integral]

Distributions on manifolds

$V(\mathcal{M})$ vector bundle, $V^\times(\mathcal{M}) := V^*(\mathcal{M}) \otimes \bigwedge^{\dim(\mathcal{M})} T^*(\mathcal{M})$ its de Rham dual.
 $V^{\times \times}(\mathcal{M}) \equiv V(\mathcal{M})$.

Correspondingly: \mathbb{F}^\times and \mathbb{F}_T^\times are de Rham duals of \mathbb{F} and \mathbb{F}_T .

$\mathbb{F} \times \mathbb{F}_T^\times \rightarrow \mathbb{R}$, $(\delta\psi, p_T) \mapsto \int_{\mathcal{M}} \delta\psi p_T$ and $\mathbb{F}_T \times \mathbb{F}^\times \rightarrow \mathbb{R}$, $(\delta\psi_T, p) \mapsto \int_{\mathcal{M}} \delta\psi_T p$ jointly sequentially continuous.

Therefore, $\mathbb{F} \subset (\mathbb{F}_T^\times)^*$ and $\mathbb{F}_T \subset (\mathbb{F}^\times)^*$ continuous dense injections in $\mathcal{E} \rightarrow \mathcal{D}^*$ and $\mathcal{D} \rightarrow \mathcal{E}^*$ sense.

(distribution valued sections)

Let $A : \mathbb{F} \rightarrow \mathbb{F}$ continuous linear.

It has formal transpose iff there exists $A^t : \mathbb{F}_T^\times \rightarrow \mathbb{F}_T^\times$ continuous linear, such that

$$\forall \delta\psi \in \mathbb{F} \text{ and } p_T \in \mathbb{F}_T^\times : \int_{\mathcal{M}} (A \delta\psi) p_T = \int_{\mathcal{M}} \delta\psi (A^t p_T).$$

Topological transpose of formal transpose $(A^t)^* : (\mathbb{F}_T^\times)^* \rightarrow (\mathbb{F}^\times)^*$ is the distributional extension of A .

Not always exists.

Fundamental solution on manifolds

Let $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}$ be Euler-Lagrange functional, and $J \in \mathbb{F}_T^*$.

$K_{(J)} \in F$ is **solution with source J** , iff $\forall \delta\psi_T \in \mathbb{F}_T : (E(K_{(J)}) | \delta\psi_T) = (J | \delta\psi_T)$.

Specially: one can restrict to $J \in \mathbb{F}_T^\times \subset \mathbb{F}^\times \subset \mathbb{F}_T^*$.

A continuous map $K : \mathbb{F}_T^\times \rightarrow F$ is **fundamental solution**, iff for all $J \in \mathbb{F}_T^\times$ the field $K(J) \in F$ is solution with source J .

May not exists, and if does, may not be unique.

If $K_{\psi_0} : \mathbb{F}_T^\times \rightarrow \mathbb{F}$ vectorized fundamental solution is linear (e.g. for linear $E_{\psi_0} : \mathbb{F} \rightarrow \mathbb{F}_T^*$):
 $K_{\psi_0} \in \mathcal{Lin}(\mathbb{F}_T^\times, \mathbb{F}) \subset (\mathbb{F}_T^\times)^* \otimes (\mathbb{F}_T^\times)^*$ is distribution.

Generalization of convolution to manifolds

A continuous linear map $C : (\mathbb{F}^\times)^* \rightarrow \mathbb{F}$ is called **smoothing operator**.

Schwartz kernels theorem: $C \longleftrightarrow \kappa \in \mathbb{F} \boxtimes \mathbb{F}^\times$, i.e. κ is field over $\mathcal{M} \times \mathcal{M}$.

$C_\kappa : \mathbb{F}_T \rightarrow \mathbb{F}$ cont.lin., where $(C_\kappa \delta\psi_T)(x) := \int_{y \in \mathcal{M}} \kappa(x, y) \delta\psi_T(y) \quad \forall \delta\psi_T \in \mathbb{F}_T, x \in \mathcal{M}$.

$C_\kappa^t : \mathbb{F}_T^\times \rightarrow \mathbb{F}^\times$ cont.lin., where $(C_\kappa^t p_T)(y) := \int_{x \in \mathcal{M}} p_T(x) \kappa(x, y) \quad \forall p_T \in \mathbb{F}_T^\times, y \in \mathcal{M}$.

partially compactly supported κ : $\forall \mathcal{K} \subset \mathcal{M}$ compact

$\{(x, y) \in \mathcal{M} \times \mathcal{M} \mid x \in \mathcal{K}, \kappa(x, y) \neq 0\}$ and $\{(x, y) \in \mathcal{M} \times \mathcal{M} \mid y \in \mathcal{K}, \kappa(x, y) \neq 0\}$ has compact closure.

Then C_κ cont.lin. $\mathbb{F} \rightarrow \mathbb{F}$ and C_κ^t cont.lin. $\mathbb{F}^\times \rightarrow \mathbb{F}^\times$.

Then C_κ cont.lin. $\mathbb{F}_T \rightarrow \mathbb{F}_T$ and C_κ^t cont.lin. $\mathbb{F}_T^\times \rightarrow \mathbb{F}_T^\times$.

On distributions, the transpose of these.

E.g. ordinary convolution over Minkowski:

\mathcal{M} (affine space), T (subordinate vector space), ν (constant volume form).

Let $\eta : T \rightarrow \mathbb{R}$ smooth compactly supported.

Let $(x, y) \mapsto \kappa(x, y) := \eta(x - y) \nu(y) I$.

Then $\delta\psi \in \mathbb{F}$: $C_\kappa \delta\psi = \eta \star \delta\psi$.

Particular solutions to the free MDS equation

Distributional solutions to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$\begin{aligned} K_{\psi_0}^{(0)} &= 0, \\ K_{\psi_0}^{(1)} &= 0, \\ K_{\psi_0}^{(2)} &= i \hbar K_{\psi_0}^{(2)} \\ K_{\psi_0}^{(n)} &= 0 \quad (n \geq 2) \end{aligned}$$

Smooth function solutions to free regularized MDS equation: $G_{\psi_0} = \exp(K_{\psi_0, \kappa})$ where

$$\begin{aligned} K_{\psi_0, \kappa}^{(0)} &= 0, \\ K_{\psi_0, \kappa}^{(1)} &= 0, \\ K_{\psi_0, \kappa}^{(2)} &= i \hbar (C_\kappa \otimes C_\kappa) K_{\psi_0}^{(2)} \\ K_{\psi_0, \kappa}^{(n)} &= 0 \quad (n \geq 2) \end{aligned}$$

[Here $C_\kappa(\cdot) := \eta \star (\cdot)$ is convolution by a test function η .]

Renormalization from functional analysis p.o.v.

Let \mathbb{F} and \mathbb{G} real or complex top.vector space, Hausdorff loc.conv complete.

Let $M : \mathbb{F} \rightarrow \mathbb{G}$ densely defined linear map (e.g. MDS operator).

Closed: the graph of the map is closed.

Closable: there exists linear extension, such that its graphs closed (unique if exists).

Closable \Leftrightarrow where extendable with limits, it is unique.

Multivalued set:

$\text{Mul}(M) := \{y \in \mathbb{G} \mid \exists (x_n)_{n \in \mathbb{N}} \text{ in } \text{Dom}(M) \text{ such that } \lim_{n \rightarrow \infty} x_n = 0 \text{ and } \lim_{n \rightarrow \infty} Mx_n = y\}.$

$\text{Mul}(M)$ always closed subspace.

Closable $\Leftrightarrow \text{Mul}(M) = \{0\}.$

Maximally non-closable $\Leftrightarrow \text{Mul}(M) = \overline{\text{Ran}(M)}.$ Pathological, not even closable part.

Polynomial interaction term of MDS operator maximally non-closable!

MDS operator:

$$\mathbf{M} : \mathbb{F}_T \otimes \mathcal{T}(\mathbb{F}) \rightarrow \mathcal{T}(\mathbb{F}_{\mathbb{C}}), \quad G \mapsto \mathbf{M} G$$

linear, everywhere defined continuous on $\mathcal{D} \otimes \mathcal{E} \rightarrow \mathcal{E}$. So,

$$\mathbf{M} : \mathcal{T}(\mathbb{F}_T^{\times *}) \hookrightarrow \mathbb{F}_T^* \otimes \mathcal{T}(\mathbb{F}_T^{\times *}), \quad G \mapsto \mathbf{M} G$$

linear, densely defined as $\mathcal{D}^* \hookrightarrow \mathcal{D}^* \otimes \mathcal{D}^*$.

Similarly: \mathbf{M}_{κ} regularized MDS operator (κ : a fix regularizator).

Not good equation:

$$G \in \mathcal{T}(\mathbb{F}_T^{\times *}) ? \quad G^{(0)} = 1 \quad \text{and} \quad \exists G_\kappa \rightarrow G \text{ approximator sequence, such that :}$$

$$\lim_{\kappa \rightarrow \delta} \mathbf{M} G_\kappa = 0.$$

All G would be selected, because $\text{Mul}()$ set of interaction term is full space.

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All G would be selected, because $\text{Mul}()$ set of interaction term is full space.

Can be good:

$$G \in \mathcal{T}(\mathbb{F}_T^{\times *}) ? \quad G^{(0)} = 1 \quad \text{and} \quad \exists G_\kappa \rightarrow G \text{ approximator sequence, such that :}$$

$$\forall \kappa : \mathbf{M}_\kappa G_\kappa = 0.$$

That is, as implicit function of κ , not as operator closure kernel.

Running coupling:

If in \mathbf{M}_κ EL terms are combined with weights $g(\kappa)$.

(Not just with real factors.)

E.g.:

$$(g, G) \in \mathcal{T}(\mathbb{F}_T^{\times *}) ? \quad G^{(0)} = 1 \quad \text{and} \quad \exists G_\kappa \rightarrow G \text{ approximator sequence, such that :}$$

$$\forall \kappa : \mathbf{M}_{g(\kappa), \kappa} G_\kappa = 0.$$

Feynman integral “ \Longleftrightarrow ” MDS equation.

Wilsonian regularized Feynman integral:

integrate not on \mathbb{F} , only on the image space $C_\kappa[\mathbb{F}]$ of a smoothing operator $C_\kappa : \mathbb{F} \rightarrow \mathbb{F}$.

[Smoothing operator: \sim convolution, can be generalized to manifolds. Does UV damping.]

Automatically knows ERGE relations.

Wilsonian regularized Feynman integral “ \Longleftrightarrow ” regularized MDS equation + ERGE:

$$(\psi_0, \kappa \mapsto g(\kappa), \kappa \mapsto G_{\psi_0, \kappa}) = ? \text{ such that : } \underbrace{G_{\psi_0, \kappa}^{(0)}}_{=: b G_{\psi_0, \kappa}} = 1,$$

$$\forall \kappa : \forall \delta\psi_T \in \mathbb{F}_T : \underbrace{\left(\mathcal{L}_{g(\kappa)}(\mathbf{E}_{\psi_0} | \delta\psi_T) - i \hbar L_{C_\kappa} \delta\psi_T \right) G_{\psi_0, \kappa}}_{=: \mathbf{M}_{\psi_0, \kappa, \delta\psi_T}} = 0,$$

$$\text{ERGE} \longrightarrow \forall \kappa', \kappa : (\otimes^n C_{\kappa'}) G_{\psi_0, \kappa}^{(n)} = G_{\psi_0, (C_{\kappa'} \kappa)}^{(n)}.$$

Running coupling is meaningful. Conjecture: ERG flow of $G_{\psi_0, \kappa} \leftrightarrow$ distributional \mathbf{G}_{ψ_0} .

(Conjecture proved for flat spacetime for bosonic fields.)