Quantification of the GR contribution to the muon g-2 measurement

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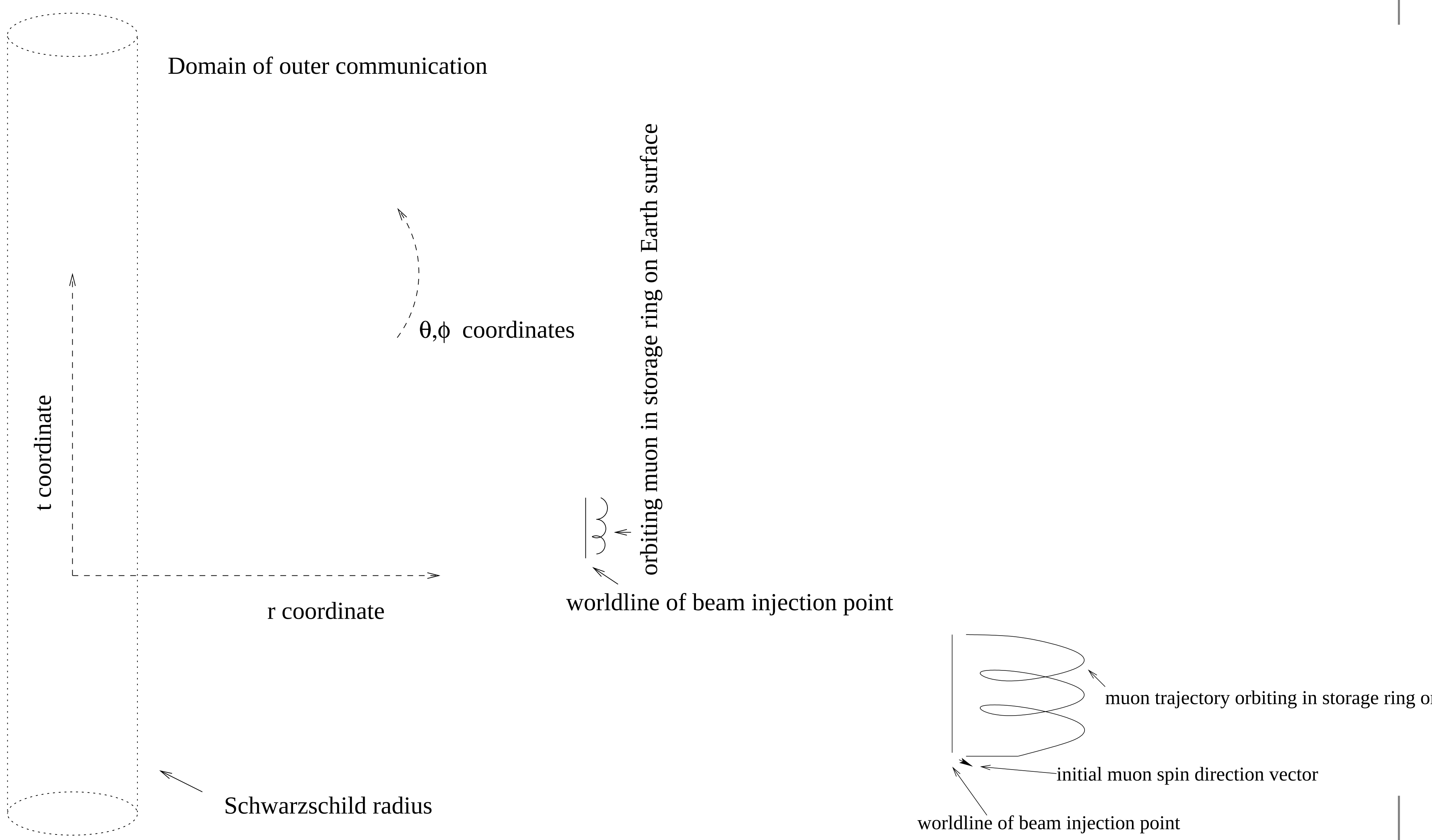
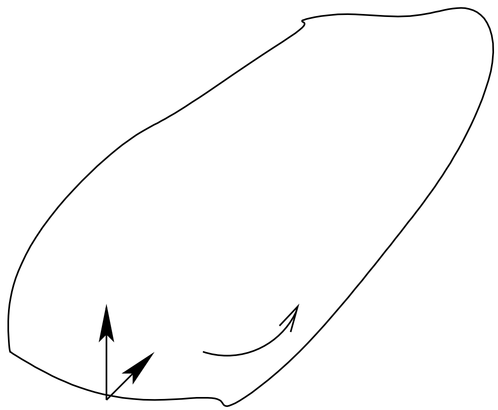
Introduction

- Recently a series of preprints appeared claiming that the General Relativity may give contribution to the muon $g-2$ experiments.
  T. Morishima, T. Futamase, H. M. Shimizu:
  `arXiv:1801.10244`, `arXiv:1801.10245`, `arXiv:1801.10246`.
- Other authors claim that the effect is very small, much beyond the experimental resolution of 10^{-7} relative error.
  M. Visser: `arXiv:1802.00651`, P. Guzowski: `arXiv:1802.01120`.
- There are also other authors, who claim that the effect exactly cancels.
  H. Nikolic: `arXiv:1802.04025`.
- Which claims hold?
- It seems to be difficult to say something from first principles.
  (Also, formulas for the de Sitter and Lense-Thirring effect do not apply.)
- One info from first principles: only Earth's field matters.
  Because the system is free falling against other gravitating objects.

The kinematic setting



Due to transporting vectors on closed loops, the effect is very likely there.

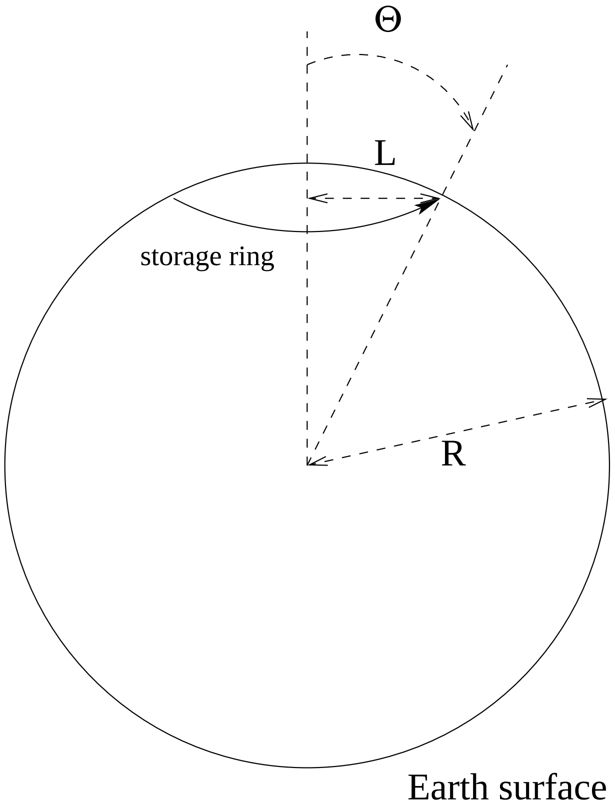


difference is expected due to curvature

Curvature is just measures this effect, but could be small.
(“Wilson loop”)

The only question seems to be: can such effect be large?

Coordinate conventions:



L=R sin Θ

Schwarzschild metric is time translationally (t) and spherically (ϑ, φ) symmetric.

Earth: $r = R = const$.

Storage ring: $r = R = const$, $\vartheta = \Theta = const$.

Elapse of time: we measure it via Killing time t.

Worldline of orbiting muon:

$$\gamma_\omega{}^{\mathsf{a}}(t) \quad = \quad \begin{pmatrix} t \\ R \\ \Theta \\ \frac{\omega}{\sqrt{1-\frac{r_S}{R}}}\, t \bmod 2\pi \end{pmatrix}$$

helical in spacetime.

Worldline of the beam injection point in the lab:

$$\gamma_0{}^{\mathsf{a}}(t) \quad = \quad \begin{pmatrix} t \\ R \\ \Theta \\ 0 \end{pmatrix}$$

a Killing time translation worldline.

The two intersect at revolution times $t = n \frac{2\pi}{\omega/\sqrt{1-\frac{r_S}{R}}}$.

In these points, transported vectors could be compared.

The relativistic gyroscope equation

How spin direction vector evolves when forced on a worldline?
It is described by the *Fermi-Walker transport*:
Hawking-Ellis: *Large Scale Structure of Spacetime*; Cambridge University Press (1973).

It is a modified version of the parallel transport ∇, such that it conserves angles.

If u^a is a future directed unit timelike vector field, the F.-W. derivative of a vector field w^b is:

$$D_u^F w^b = u^d \nabla_d w^b + g_{ac} w^a u^b u^d \nabla_d u^c - g_{ac} w^a u^c u^d \nabla_d u^b$$

Properties:
(i) $D_u^F u^b = 0$, (ii) whenever $D_u^F w^b = 0$ and $D_u^F v^b = 0$ then $u^d \nabla_d (w_b v^b) = 0$ follows.

Relativistic gyroscope equation:

$$D_u^F w^b = 0$$

It is a parallel transport, taking into account the constraints

$$u_a u^b = 1, \quad w_a w^b = -1, \quad u_a w^b = 0.$$

See also: Gravity Probe B satellite experiment.

We will just try to solve gyroscope equation for the $g-2$ situation.
(Now we do not calculate electromagnetic precession.)
Full equation would be *Bargmann-Michel-Telegdi equation*:

$$\begin{aligned} u^a \nabla_a u^b &= \frac{e}{m} F^{ab} u_b \\ D_u^F w^b &= 2\,\mu \left(F^{bc} - u^b u_d F^{dc} \right) w_c, \end{aligned}$$

which causes *Larmor precession*.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} w_\omega{}^{\mathsf{b}}(t) \;+\; \dot\gamma_\omega{}^{\mathsf{d}}\, \Gamma^{\mathsf{b}}_{\mathsf{dc}}\, w_\omega{}^{\mathsf{c}}(t) \\
&+ g_{\mathsf{ac}} w_\omega{}^{\mathsf{a}}(t) \frac{1}{{\Lambda_\omega}^2} \dot\gamma_\omega{}^{\mathsf{b}} \dot\gamma_\omega{}^{\mathsf{d}} \Gamma^{\mathsf{c}}_{\mathsf{de}} \dot\gamma_\omega{}^{\mathsf{e}} \;-\; g_{\mathsf{ac}} w_\omega{}^{\mathsf{a}}(t) \frac{1}{{\Lambda_\omega}^2} \dot\gamma_\omega{}^{\mathsf{c}} \dot\gamma_\omega{}^{\mathsf{d}} \Gamma^{\mathsf{b}}_{\mathsf{de}} \dot\gamma_\omega{}^{\mathsf{e}} \quad = \quad 0, \\
&\frac{\mathrm{d}}{\mathrm{d}t} w_0{}^{\mathsf{b}}(t) \;+\; \dot\gamma_0{}^{\mathsf{d}}\, \Gamma^{\mathsf{b}}_{\mathsf{dc}}\, w_0{}^{\mathsf{c}}(t) \\
&+ g_{\mathsf{ac}} w_0{}^{\mathsf{a}}(t) \frac{1}{{\Lambda_0}^2} \dot\gamma_0{}^{\mathsf{b}} \dot\gamma_0{}^{\mathsf{d}} \Gamma^{\mathsf{c}}_{\mathsf{de}} \dot\gamma_0{}^{\mathsf{e}} \;-\; g_{\mathsf{ac}} w_0{}^{\mathsf{a}}(t) \frac{1}{{\Lambda_0}^2} \dot\gamma_0{}^{\mathsf{c}} \dot\gamma_0{}^{\mathsf{d}} \Gamma^{\mathsf{b}}_{\mathsf{de}} \dot\gamma_0{}^{\mathsf{e}} \quad = \quad 0 \quad (t \in \mathbb{R}).
\end{aligned}
$$

Because of the symmetries of the spacetime and of the orbit, this is a homogeneous linear differential equation with *constant coefficients*!

$$
\begin{aligned}
\Lambda_\omega &:= \sqrt{g_{\mathsf{ab}} \dot\gamma_\omega{}^{\mathsf{a}} \dot\gamma_\omega{}^{\mathsf{b}}}, \\
\Lambda_0 &:= \sqrt{g_{\mathsf{ab}} \dot\gamma_0{}^{\mathsf{a}} \dot\gamma_0{}^{\mathsf{b}}}
\end{aligned}
$$

in order to compensate Killing time $\leftrightarrow$ proper time.

Actually, the Fermi-Walker transport equation of any vector ${v_0}^b$ along the lab reference curve $(t \mapsto \gamma_0(t))$ simplifies to $\frac{\mathrm{d}}{\mathrm{d}t}{v_0}^b(t) = 0$.

Thus, the second equation simplifies:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}{w_\omega}^{\mathsf{b}}(t) \;+\; {\dot\gamma_\omega}^{\mathsf{d}}\,\Gamma^{\mathsf{b}}_{\mathsf{dc}}\,{w_\omega}^{\mathsf{c}}(t) & \\
+\, g_{\mathsf{ac}}{w_\omega}^{\mathsf{a}}(t)\frac{1}{{\Lambda_\omega}^2}{\dot\gamma_\omega}^{\mathsf{b}}{\dot\gamma_\omega}^{\mathsf{d}}\Gamma^{\mathsf{c}}_{\mathsf{de}}{\dot\gamma_\omega}^{\mathsf{e}} \;-\; g_{\mathsf{ac}}{w_\omega}^{\mathsf{a}}(t)\frac{1}{{\Lambda_\omega}^2}{\dot\gamma_\omega}^{\mathsf{c}}{\dot\gamma_\omega}^{\mathsf{d}}\Gamma^{\mathsf{b}}_{\mathsf{de}}{\dot\gamma_\omega}^{\mathsf{e}} &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}t}{w_0}^{\mathsf{b}}(t) &= 0 \quad (t \in \mathbb{R}).
\end{aligned}$$

Homogeneous linear differential equations with constant coefficients!

Evolution equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} w_\omega{}^{\mathsf{b}}(t) = \mathcal{F}_\omega{}^{\mathsf{b}}{}_{\mathsf{a}}\, w_\omega{}^{\mathsf{a}}(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t} w_0{}^{\mathsf{b}}(t) = 0 \quad (t \in \mathbb{R}).$$

Analytic expression of the evolution matrix:

$$\mathcal{F}_\omega{}^{\mathsf{b}}{}_{\mathsf{c}} = \frac{\omega}{\sqrt{1-\frac{r_S}{R}}}\,\frac{1}{1-\frac{\omega^2 L^2}{\left(1-\frac{r_S}{R}\right)^2}}$$

$$\begin{pmatrix} 0 & -\omega L\,\frac{L}{R}\,\frac{1-\frac{3}{2}\,\frac{r_S}{R}}{\left(1-\frac{r_S}{R}\right)^{\frac{5}{2}}} & -\omega L\,\frac{R}{\left(1-\frac{r_S}{R}\right)^{\frac{3}{2}}}\sqrt{1-\left(\frac{L}{R}\right)^2} & 0 \\ -\omega L\,\frac{L}{R}\,\frac{1-\frac{3}{2}\,\frac{r_S}{R}}{\left(1-\frac{r_S}{R}\right)^{\frac{1}{2}}} & 0 & 0 & L\,\frac{L}{R}\,\frac{1-\frac{7}{2}\,\frac{r_S}{R}+4\left(\frac{r_S}{R}\right)^2-\frac{3}{2}\left(\frac{r_S}{R}\right)^3}{\left(1-\frac{r_S}{R}\right)^2} \\ -\omega L\,\frac{1}{R}\,\frac{1}{\left(1-\frac{r_S}{R}\right)^{\frac{1}{2}}}\sqrt{1-\left(\frac{L}{R}\right)^2} & 0 & 0 & \frac{L}{R}\sqrt{1-\left(\frac{L}{R}\right)^2} \\ 0 & -\frac{1}{R}\,\frac{1-\frac{3}{2}\,\frac{r_S}{R}}{\left(1-\frac{r_S}{R}\right)} & -\frac{R}{L}\sqrt{1-\left(\frac{L}{R}\right)^2} & 0 \end{pmatrix}$$

$\mathcal{F}_\omega{}^{\mathsf{b}}{}_{\mathsf{c}} g^{\mathsf{ca}}$ is antisymmetric, thus $\mathcal{F}_\omega{}^{\mathsf{b}}{}_{\mathsf{a}}$ describes a Lorentz transformation generator. Also, $\mathcal{F}_\omega{}^{\mathsf{b}}{}_{\mathsf{a}}\ u_\omega{}^{\mathsf{a}} = 0$, thus it is an u_ω-rotation. (u_ω, u_0 are four velocities of γ_ω, γ_0.)

This is an absolute, observer independent effect, called **Thomas rotation**.

Several analytic crosschecks were performed.

- The evolving matrix for $\omega = 0$ must vanish.
- The matrix evolving $\dot{\gamma}_\omega$ must annihilate $\dot{\gamma}_\omega$.
  (Because $\dot{\gamma}_\omega$ is constant in t.)
- Conserved angles: $w_a \mathcal{F}_\omega{}^a{}_b w^b = 0$ etc.
- GRTensorII used.

Crosschecks passed analytically!

Actually, one can extract the angular velocity vector of the Thomas rotation effect by taking the Hodge dual of the tensor $\mathcal{F}_\omega$, in the comoving frame u_ω, and converting Killing time to u_ω-proper time.

$$\Omega^{u_\omega}_{\omega,r_S}{}^{\mathsf{f}} := u_{\omega,r_S}{}^{\mathsf{a}} \sqrt{-\det(g_{r_S})}\, \epsilon_{\mathsf{abcd}} g_{r_S}{}^{\mathsf{bf}} \frac{1}{\Lambda_\omega} \mathcal{F}_{\omega,r_S}\, {}_e{}^c g_{r_S}{}^{ed}$$

One can then calculate its length to extract the angular velocity of the Thomas rotation:

$$|\Omega|^{u_\omega}_{\omega,r_S} := \sqrt{-g_{r_S\,\mathsf{ab}}\, \Omega^{u_\omega}_{\omega,r_S}{}^{\mathsf{a}} \Omega^{u_\omega}_{\omega,r_S}{}^{\mathsf{b}}}$$

Has first order GR correction!

$$\frac{r_S \left(\left. \frac{\mathrm{d}\, |\Omega|^{u_\omega}_{\omega,r_S}}{\mathrm{d} r_S} \right|_{r_S=0} \right)}{\left(\left. |\Omega|^{u_\omega}_{\omega,r_S} \right|_{r_S=0} \right)} = \frac{r_S}{R} \gamma^2 \left((1+\beta^2) - \left(\frac{L}{R}\right)^2 \frac{1}{\gamma^2} \right)$$

Here, $\beta := L\omega$, $\gamma := \frac{1}{\sqrt{1-\beta^2}}$. Ultrarelativistic limit ($\beta \to 1$): $\approx \frac{r_S}{R} 2\gamma^2$.

One can convert the effect to the Einstein synchronized laboratory observer u_0.

$$\left. |\Omega|^{u_0}_{\omega,r_S} \right|_{r_S=0} = \omega \frac{\beta^2\gamma^2}{1+\gamma}$$

(textbook formula for special relativistic **Thomas precession**).

Has first order GR correction! In the $\beta \to 1$ limit:

$$\frac{r_S \left(\left. \frac{\mathrm{d}\,|\Omega|^{u_0}_{\omega,r_S}}{\mathrm{d}r_S} \right|_{r_S=0} \right)}{\left(\left. |\Omega|^{u_0}_{\omega,r_S} \right|_{r_S=0} \right)} = \frac{r_S}{R} \frac{\gamma^3}{1+\gamma} \approx \frac{r_S}{R}\gamma^2$$

$r_S \approx 9 \cdot 10^{-3}\,\mathrm{m}$, $R \approx 6.371 \cdot 10^6\,\mathrm{m}$, $\gamma \approx 29.3$, the relative error term quantifies as:

$$1.17 \cdot 10^{-6}.$$

Above the planned experimental accuracy of the $g-2$ experiment ($\approx 10^{-7}$).