Truncated spectrum methods

and the self-duality of the sinh-Gordon model

Based on Konik, Lájer, Mussardo, arXiv:2007.00154

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Contents

Introduction

Quantum field theory (QFT)

- Framework for describing the fundamental interactions of Nature: At least 3 out of 4 (weak, EM, strong)
- Effective models in particle, statistical and solid state physics
- In most cases, only approximate solutions available

INPUT $\mathcal{D}X_i$ Boundary conditions $S[X_i] = \int d^D x \mathcal{L}(X_i, \partial_\mu X_i)$

$$Z[J_i] = \int \mathcal{D}X_i e^{\frac{i}{\hbar}S[x_i] + J_i X_i}$$

OUTPUT Particle masses S matrix operator matrix elements n-point functions etc.

QFT= Quantum mechanics + Poincaré invariance

$\mathcal{H}$ **Hilbert space**

Finite speed information propagation

States: Time evolution: $U(t) = e^{-iHt}$

H: Continuum spectrum

Ground state (vacuum) $|\mathrm{vac}\rangle$ $E_{\mathrm{vac}} = E_0$

One particle states $\left|p^{(i)}\right\rangle$ $E_1 = E_0 + M_i \cosh\vartheta$

Two particle states $\left|p_1^{(i)}, p_2^{(j)}\right\rangle$ $E_2 = E_0 + M_i \cosh\vartheta_1 + M_j \cosh\vartheta_2$

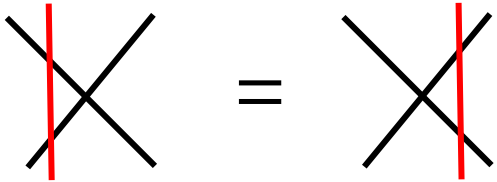
...

Measurements: **D=4** (Universe)

Significance of **D<4**

- Experimentally relevant (e.g. anisotropic materials, trapped quantum gases)
- Toy models (many exact results)
- String theory (Background: D=10/11/26, but string worldsheet is D=2)

D=2(=1+1): integrability



Higher spin conserved charges → P-dependent translation invariance. → **Factorised S-mátrix**

$$S_{ij}^{kl}(\vartheta_i - \vartheta_j)$$

- Dynamics completely determined by 2->2 S-matrices
- (Almost) exclusively completely elastic scattering
- At most particle type can change, but not its rapidity

Finite volume L (1 particle type):

$$S(\vartheta) = e^{i\delta(\vartheta)}$$

$$j \in \{1..n\}: \qquad e^{ip_jL}\prod_{j\neq k}^{n} S(\vartheta_j - \vartheta_k) = 1$$

Wave function periodic → **Discrete spectrum!**

$$Q_j^0(\{\vartheta\}) \equiv p_jL + \sum_{j\neq k}^{n} \delta(\vartheta_j - \vartheta_k) = 2\pi I_j, \qquad I_j \in \mathbb{Z}$$

Bethe-Yang equation
Polynomial corrections in 1/L

$$Q_j(\{\vartheta\}) \equiv Q_j^0(\{\vartheta\}) + \delta Q(\{\vartheta\}) = 2\pi I_j$$

Excited state TBA momentum quantization
Exact, $O(e^{-ML})$ corrections summed up

Energy levels

Bethe-Yang

$$E(\{I_j\}) = \sum_{j=1}^{n} M \cosh \vartheta_j$$

Exc. TBA (exact)

$$E(\{I_j\}) = \sum_{j=1}^{n} M \cosh \vartheta_j + \delta E(\{I_j\})$$

Correlators:

$$\langle O_1(x) O_2 \rangle = \sum_{n=0}^{\infty} \int \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} \langle \mathrm{vac} \,|O(x)|\, \theta_1 \dots \theta_n \rangle_{\mathrm{in}} \langle \theta_1 \dots \theta_n \,|O(x)|\, \mathrm{vac} \rangle$$

Form factors

	Polynomial	**Exponential**
Diagonal	Pozsgay-Takács Bajnok-Wu	Leclair-Mussardo, Pozsgay-Szécsényi-Takács, Negro-Smirnov, Bajnok-Smirnov...
Non-diagonal	Pozsgay-Takács	Bajnok-Balog-Lájer-Wu, Bajnok-Lájer-Szépfalvi-Vona

Sinh-Gordon model (ShG)

Lagrangian definition (path integral)

$$\mathcal{L} = \frac{1}{16\pi}(\partial_\nu \varphi)^2 - 2\mu \cosh(b\varphi)$$

integrable

Viewpoints:

$$\mathcal{L} = \mathcal{L}_{\text{Gauss}} - 2\mu \cosh(b\varphi)$$

sine-Gordon analytic continuation

$$\mathcal{L} = \mathcal{L}_{\text{m}} - \left(\frac{m^2}{b^2}\cosh(b\varphi) - \frac{m^2}{2}\varphi^2\right)$$

1 particle, completely elastic scattering

$$\mathcal{L} = \mathcal{L}^{b}_{\text{Liouville}} - \mu e^{-b\varphi} = \mathcal{L}^{-b}_{\text{Liouville}} - \mu e^{b\varphi}$$

$b \to \frac{1}{b}, \mu \to \tilde{\mu}$ Duality in *analytically continued* quantities, Seiberg bounds

S-matrix bootstrap

$$S(\theta) = \frac{\sinh\theta - i\sin\pi B}{\sinh\theta + i\sin\pi B}$$

Exact finite volume spectrum as function of physical mass

(Relative) form factors (form factor bootstrap)

CONJECTURE:

The above S-matrix describes scattering in the lagrangian theory on the left (after renormalization), furthermore:

$$B(b) = \frac{b^2}{1+b^2}$$

Canonical quantization of Lagrangian theory:
Only **one** parameter to renormalize (normal ordering, tadpole=0)

Conform scheme: $\mu \to \mu_{\mathrm{ShG}}$

Massive scheme: $\mu \to \frac{m^2}{b^2}$

Finite volume Hamiltonian:

$$H(L) = H_0^{(\alpha)} + Z_\alpha(L)\mu_{\mathrm{ShG}} L : \cosh(b\varphi) :_{\alpha,L} \delta_P + \tilde{E}_0^{(\alpha)}(L)$$

Diagonal (unperturbed energies)

Volume dependent renormalization

Casimir energy-type correction

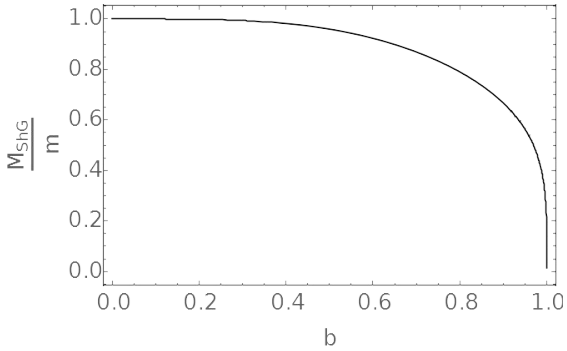
$Z_\alpha(L), \tilde{E}_0^{(\alpha)}(L)$ depend on the choice of the unperturbed part; exact determination possible if $H_0^{(\alpha)}$ Is a free model

Exact, finite relation between μ_{ShG} and M_{ShG} (well-supported conjecture):

$$\mu_{ShG} = \frac{m^{2+2b^2}}{2^{4+2b^2}\pi b^2} e^{2b^2\gamma_E}$$

$$M_{ShG} = \frac{4\sqrt{\pi}}{\Gamma\left(\frac{1}{2+2b^2}\right)\Gamma\left(1+\frac{b^2}{2+2b^2}\right)} \left[-\mu_{ShG}\frac{\pi\Gamma(1+b^2)}{\Gamma(-b^2)}\right]^{\frac{1}{2+2b^2}}$$

b>1 complex!



Similar problems with VEV: $\langle e^{a\varphi} \rangle \sim M_{\mathrm{ShG}}^{-2a^2}$

$$\langle e^{a\varphi} \rangle < 0, \quad |\alpha| > \frac{Q}{2}, \quad Q = b + b^{-1}$$

Seiberg-bounds?

Step 1: Expand H(L) to order b^4 (time-independent /old-fashioned/ perturbation theory around free massive boson

$$E_1(L) - E_0(L)$$

Excited state TBA

(Volume-exact formulae, double sum)

↓

Contour integral technique

↓

Lüscher-type integrals

$$\langle \mathrm{vac} | \varphi | p \rangle$$

Newly calculated F-term (BBLW)

Volume exact confirmation ✓

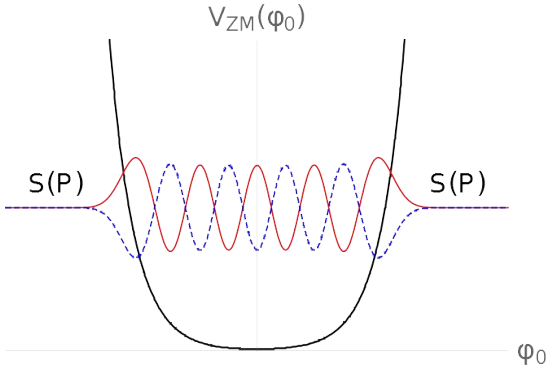
Bajnok, Balog, Lájer, Wu (2018)

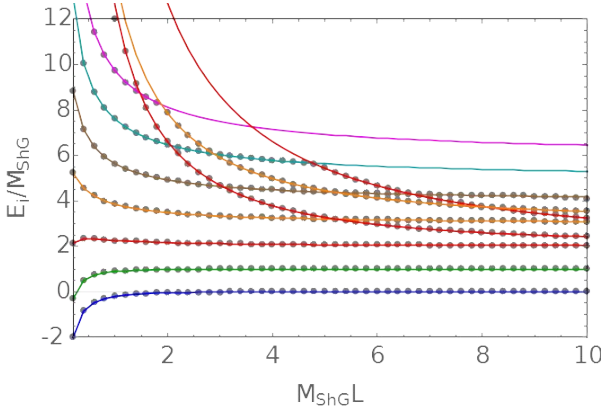
Leading Lüscher correction agrees with expansion of F-term ✓

Spectrum of H(L) is discrete: **Truncated Spectrum Method (TSM)**

$$H = H_{\mathrm{ZM}} + H_{\mathrm{osc}} + H_1 \qquad H_\Lambda \psi = E_\Lambda \psi$$

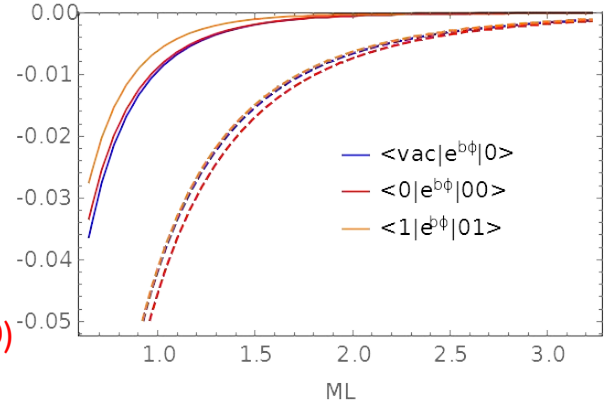
1. **Variational method**: introduce an (energy) cutoff Λ in the unperturbed Hilbert space; search eigenvectors/eigenvalues in this finite basis
2. Useful to treat quantum mechanical zero mode separately: Particle in a box of width D with cosh potential
3. Try to correct effect of cutoff Λ
4. Dependence on Λ negligible for small b





- BY equations -> identification of states
- Matrix elements -> numerical finite volume FF
- Confirmation of general F-term

Bajnok, Lájer, Szépfalvi, Vona (2019)



1. Cutoff dependence

(Massless) truncated TSM basis

$$\mathcal{H} = \left\{ a^\dagger_{n_1} \dots a^\dagger_{n_k} |0\rangle \otimes a^\dagger_{-m_1} \dots a^\dagger_{-m_l} |0\rangle \otimes |s\rangle \sum_{i=1}^{k} n_k \leq N_c \sum_{m=1}^{l} n_l \leq N_c; s = 1, \cdots, N_{ZM} \right\}$$

$$H_0 = H_{ZM} + H_{NZM}$$

$$H_{ZM} = \frac{4\pi}{L}\Pi_0^2 + \mu_{ShG} L \left(\frac{L}{2\pi}\right)^{2b^2} \left[e^{b\varphi_0} + e^{-b\varphi_0}\right]$$

$$H_{NZM} = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{1}{12}\right) \qquad L_0 = \sum_{n>0} n a^\dagger_n a_n \ , \ \bar{L}_0 = \sum_{n<0} |n| \, a^\dagger_n a_n$$

Perturbation

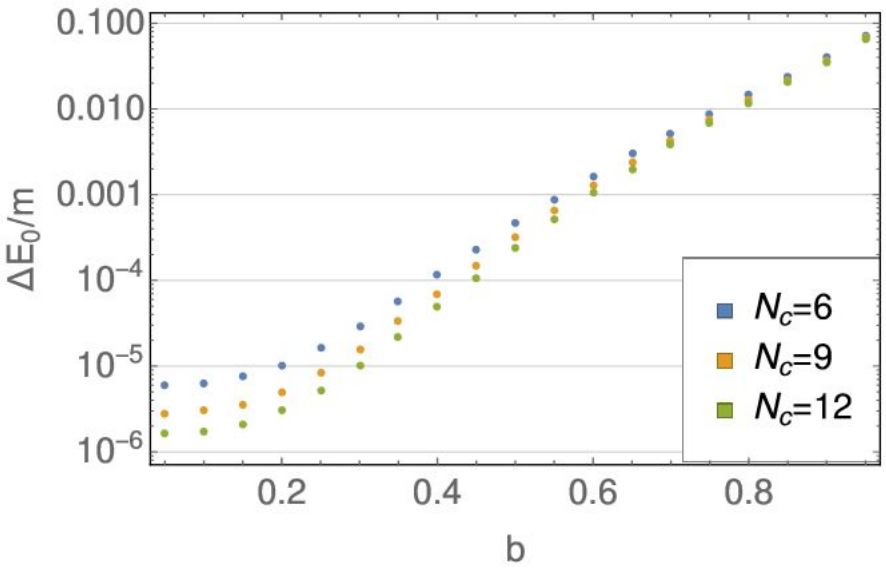
$$\int_0^L dx\, V_{pert}(x) = \delta_{P,0}\mu_{ShG}\left(\frac{L}{2\pi}\right)^{2b^2} L\left[e^{b\varphi_0}\left(:e^{b\tilde{\varphi}(0)}:-1\right)+e^{-b\varphi_0}\left(:e^{-b\tilde{\varphi}(0)}:-1\right)\right]$$

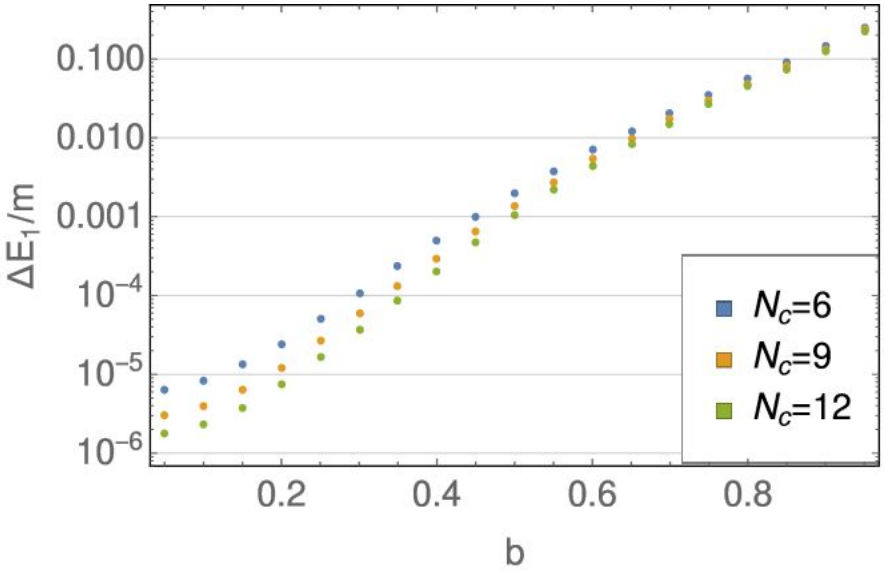
$$\varphi(x,\tau) = \varphi_0(\tau) + \tilde{\varphi}\,(x,\tau) \equiv \varphi_0(\tau) + \varphi_R(x,\tau) + \varphi_L(x,\tau) + \varphi_R^\dagger(x,\tau) + \varphi_L^\dagger(x,\tau)$$

Normal ordering: $:e^{b\varphi(x,\tau)}: \equiv e^{b\phi_0(\tau)} e^{b\varphi_R^\dagger(x,\tau)} e^{b\varphi_R(x,\tau)} e^{b\varphi_L^\dagger(x,\tau)} e^{b\varphi_L(x,\tau)}$

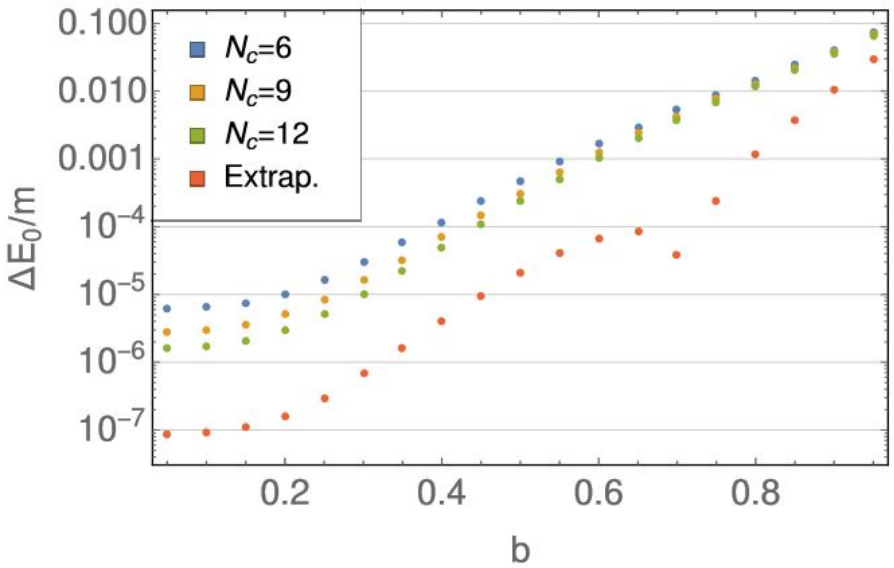
$$\varphi_R\,(x,\tau) = \sum_{n>0}\sqrt{\frac{2}{|n|}}a_n e^{ik_n x - k_n\tau} \qquad \varphi_L\,(x,\tau) = \sum_{n<0}\sqrt{\frac{2}{|n|}}a_n e^{ik_n x - |k_n|\tau}$$

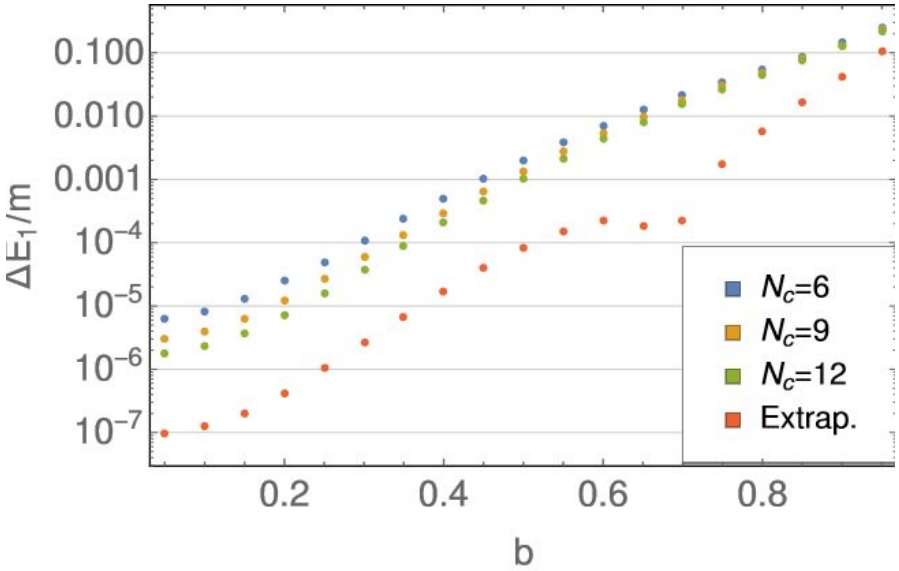
Finite volume ($M_{ShG}L$=1) energy levels



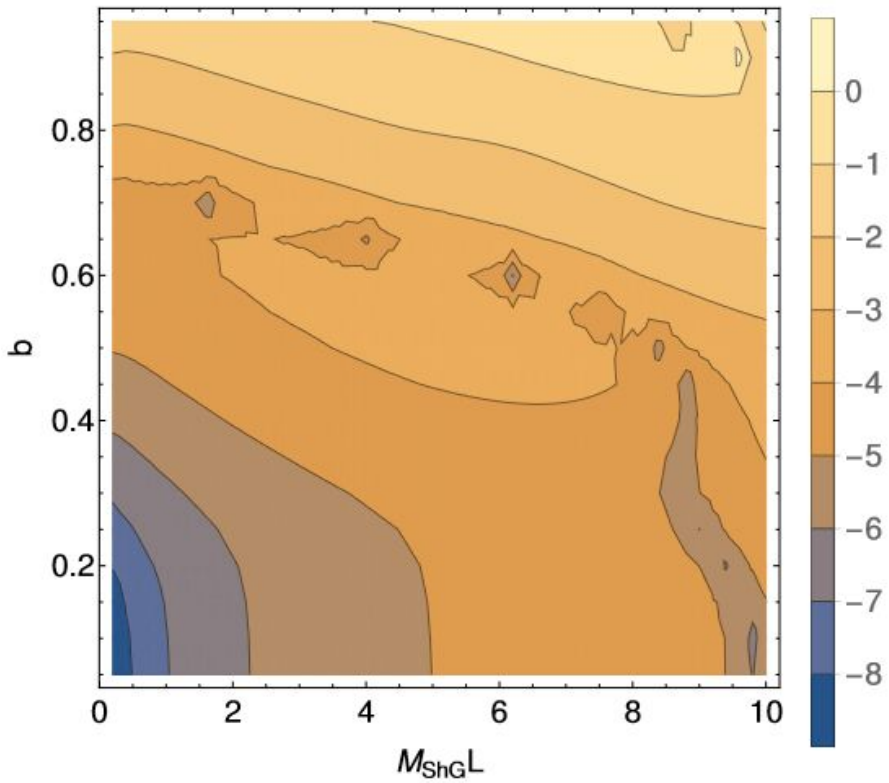


Finite volume ($M_{ShG}L$=1) energy levels

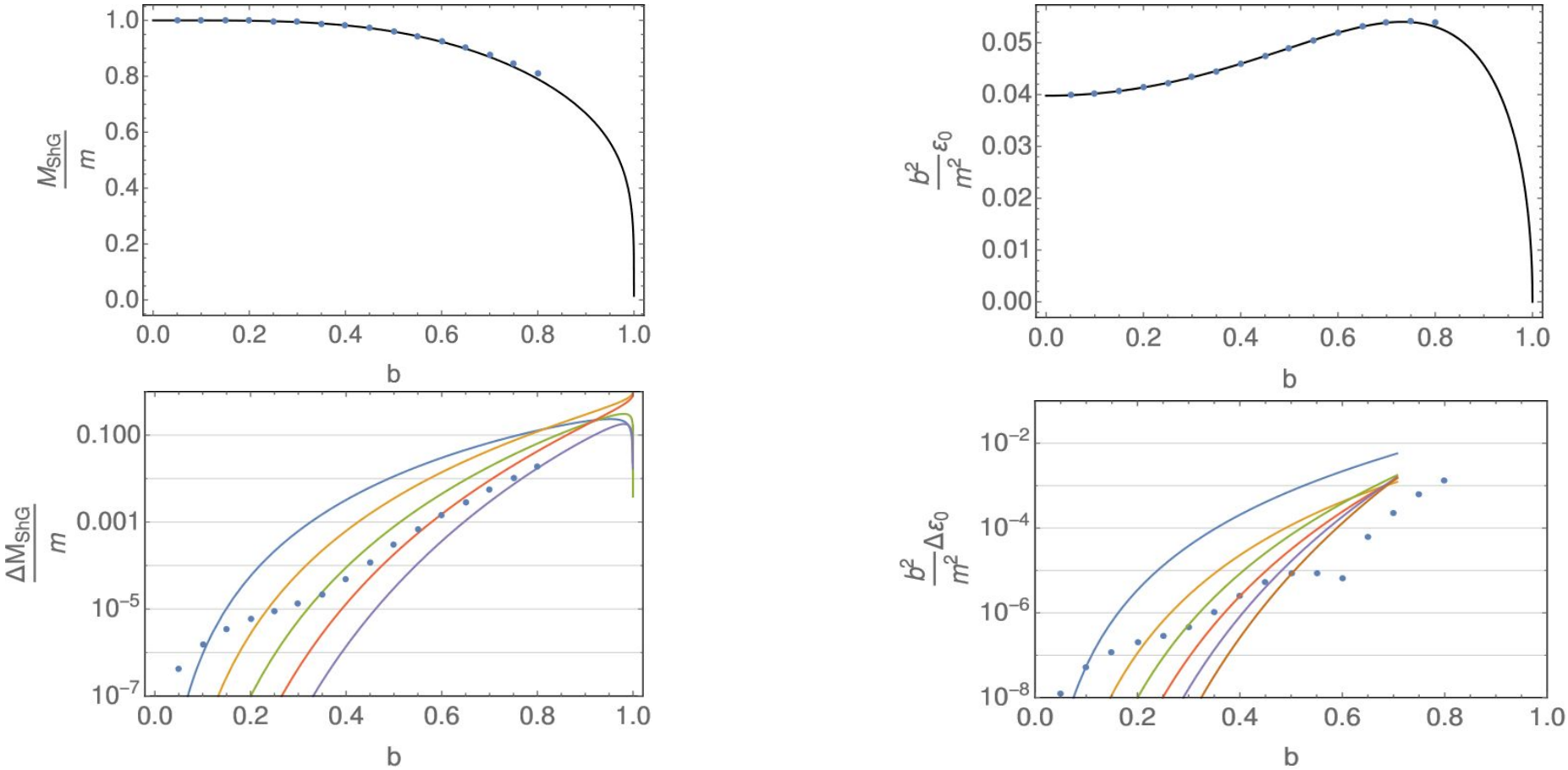




Accuracy with respect to TBA

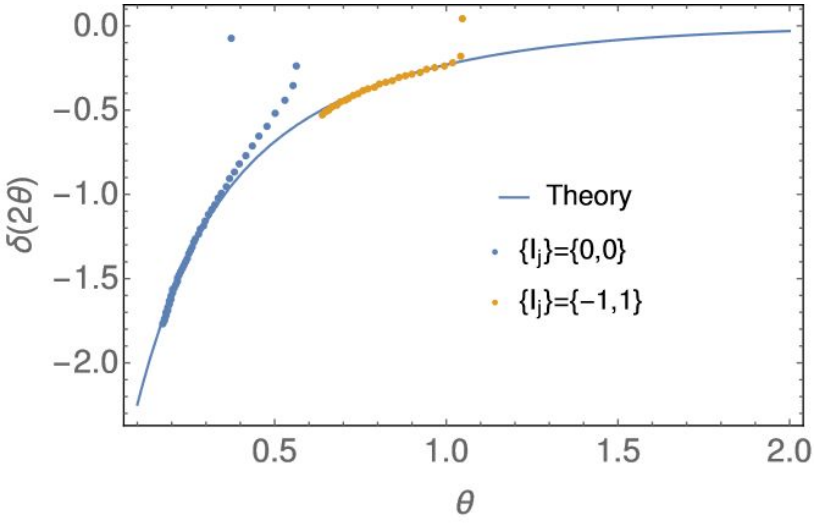


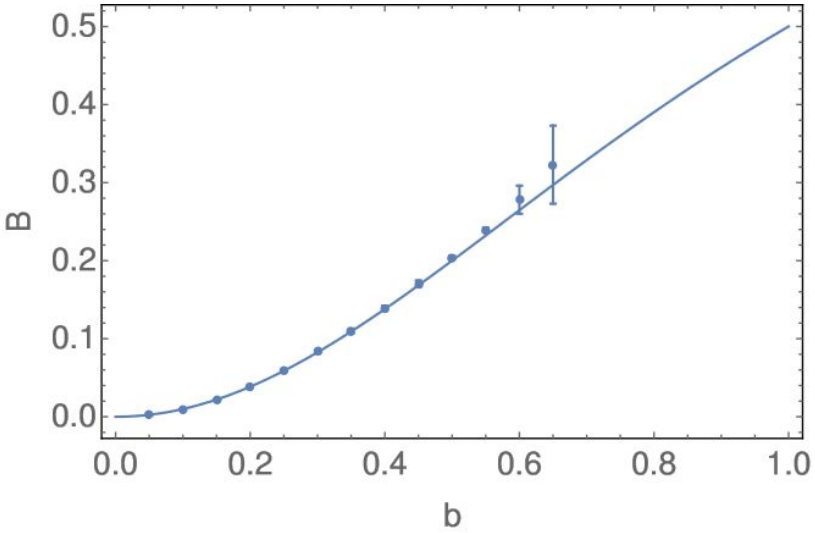
Mass and vacuum energy density



S matrix parameter B

$$S(\theta) = e^{i\delta(\theta)}$$





BY equation:

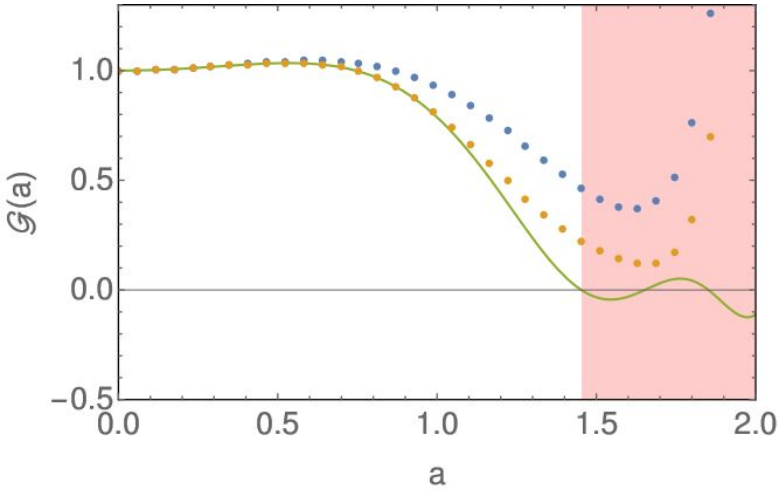
$$\delta(2\theta) + M_{ShG} L \sinh(\theta) = 2\pi n$$

Two-particle state:

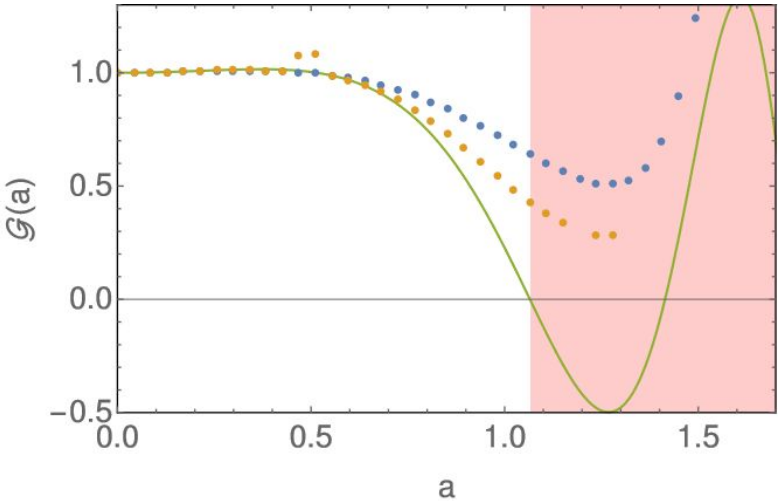
$$\theta = \mathrm{Arcosh}\left(\frac{E - E_0}{2M_{ShG}}\right)$$

Vacuum expectation values

$$\langle e^{\alpha\phi}\rangle \equiv M_{ShG}^{-2\alpha^2}\,\mathcal{G}(\alpha)$$



$$b = \frac{2}{\sqrt{8\pi}} \approx 0.4$$

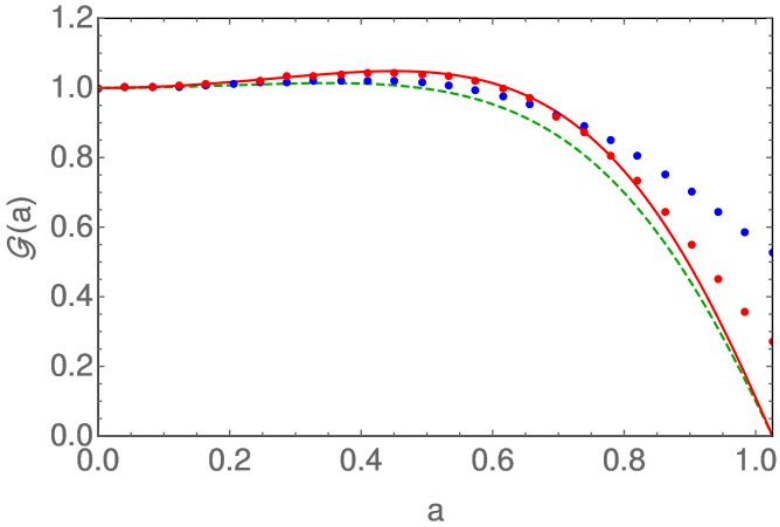


$$b = \frac{4}{\sqrt{8\pi}} \approx 0.8$$

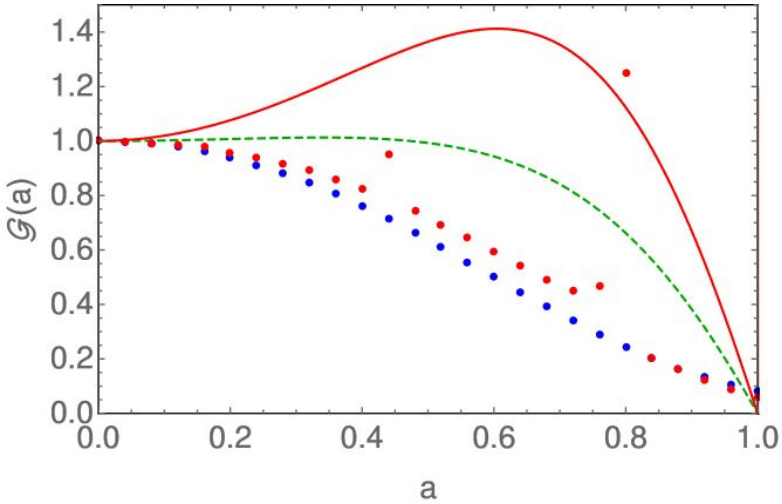
$$\mu_{ShG} = 0.1, L = 6$$

Vacuum expectation values

$$\langle e^{\alpha\phi}\rangle \equiv M_{ShG}^{-2\alpha^2}\,\mathcal{G}(\alpha)$$



$$b = \frac{4}{\sqrt{8\pi}} \approx 0.8$$

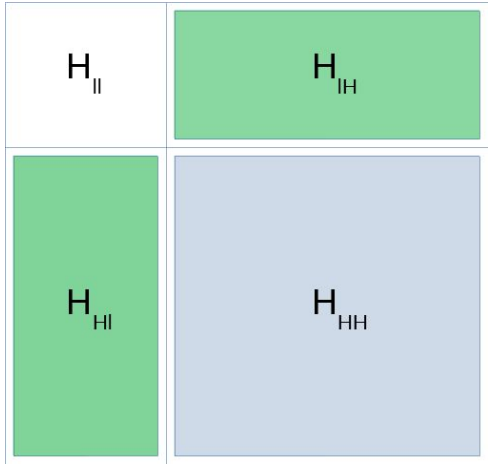


$$b = \frac{5}{\sqrt{8\pi}} \approx 0.99$$

$$\mu_{ShG} = 0.1, L = 2$$

Increasing b: Λ-dependence strengthens

- Techniques exist to decrease cutoff dependence (Pl. Lencsés-Takács, Hogervorst-Rychkov-van Rees, Elias-Miró-Rychkov-Vitale,...)
- All based on an expansion in a coupling constant
- Expansion in b: OK, but we get a Landau-Ginzburg model
- Expansion in μ_{ShG}:



$$H_{\mathrm{eff}} = H_{ll} + \delta H \qquad \delta H = H_{lH} \frac{1}{E_* - H_{HH}} H_{Hl}$$

$$\delta H = \sum_{n=2}^{\infty} \delta H_n \mu_{\mathrm{ShG}}^n$$

$$\delta H_n \propto \Lambda^{2(n^2-n)b^2-2n+2}$$

Singular for $b \geq \frac{1}{\sqrt{n}}$

All b>0: infinitely many singular terms!

$$\delta H_2 \propto \Lambda^{4b^2-2}$$

Singular for $b \geq \frac{1}{\sqrt{2}}$

TSM numerics show no trace of any of these divergences!

Toy example

$$f(\mu,\epsilon) = \int_{\epsilon}^{1} dx (x+\mu)^{\frac{1}{b^2}-1}$$

$$f(\mu,0) = b^2((1+\mu)^{\frac{1}{b^2}} - \mu^{\frac{1}{b^2}})$$

Attempt an expansion around μ=0:

$$f(\mu,\epsilon) = \sum_{k=0}^{\infty} \mu^k \binom{b^{-2}-1}{k} \int_{\epsilon}^{1} x^{b^{-2}-1-k} = \sum_{k=0}^{\infty} \mu^k \binom{b^{-2}-1}{k} \frac{1}{k-b^{-2}} (\epsilon^{b^{-2}-k}-1)$$

Singular as $\varepsilon \to 0$ for $k > b^{-2}$

Observation 1: Same type of divergences appear in expansion of energy levels

True μ-dependence of energy levels from small volume expansion of TBA:

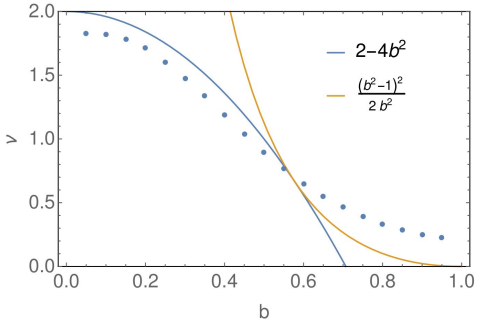
$$E_n = \frac{1}{L} \sum_k \frac{\alpha_k}{\log^k \mu} + O(\mu)$$

Observation 2: TSM data can be fitted robustly with the function $E(\Lambda) = E_{\text{extrap}} + c\Lambda^{\gamma}$

New strategy: Try to resum the series $E(\Lambda) = \sum_n \alpha_n \Lambda^{2(n^2-n)b^2-2n+2}$

for finite Λ and study the asymptotics of the resummed function. It can be shown (with reasonably weak assumptions), that the asymptotics is indeed a power-law, and the predicted exponent is

$$\gamma = \frac{(b^2-1)^2}{2b^2}$$



Extra terms to the Hamiltonian? (e.g. $\tilde{\mu} e^{\frac{1}{b}\varphi}$)

- Idea: examine the UV spectrum
- Most precise results ofTSM expected at small volumes
- L->0: Energy of oscillator states diverges, naively one expects the zero mode to dominate
- Instead, the differences between zero-mode and exact energies diverge at L->0

analytic			numeric
Semi-classical(ZM)	Exact (TBA)	Effective ZM potential (new)	TSM (new)
$S_0^2(P) = 1$	$S_L^2(P) = 1$	$S_2^2(P) = 1$	
$O(b^8)$ error		$O(b^{12})$ error	Best approximation to exact energies

Both analytical and numerical confirmation that the presence of oscillators completely explains the L->0 behavior. By all accounts there's no need of any extra terms in the (bare) potential

2. Form factor TSM

Form factor TSM

Based on what has been learned so far: construct the Hamiltonian on the basis of eigenstates of a different sinh-Gordon model (using form factors)

$$H = H_{b_0}^{(\mathrm{ShG})} + H_1$$

$$H_{b_0}^{(\mathrm{ShG})} = H_0 + 2\mu_0 \int_0^L dx \cosh b_0 \varphi$$

diagonal

$$H_1 = \int_0^L dx \left(2\mu_1 \cosh b_1 \varphi - 2\mu_0 \cosh b_0 \varphi\right)$$

Mátrix elements known (up to exponentially small corrections)

Sanity checks:

$$b_0 \to 0$$

Reproducing matrix elements of massive basis

$$b_1 = b_0 + \epsilon$$

Exact mass-coupling relation from form factor perturbation theory

Double truncation: number of particles N_{max}, momentum (quantization number) cutoff k_{max}

Basis size: up to **1174** dimensions (each parity sector)

Volume range: $5 \leq M_{\mathrm{ShG}} L \leq 15$

Exponential corrections neglected:

- ➔ Matrix elements of H0 from Bethe-Yang energies
- ➔ Matrix elements of H1: Koubek-Mussardo, Lukyanov-Zamolodchikov, Pozsgay-Takács formulae

Matrix elements

Pozsgay-Takács (2007), non-diagonal case:

$$\left\langle \{I_i\}_{i=1}^k \left|\mathcal{O}\left(x,0\right)\right| \{\tilde{I}_j\}_{j=1}^l \right\rangle_L = \frac{e^{iM\left(\sum_{j=1}^l \sinh\vartheta_j - \sum_{i=1}^k \sinh\theta_i\right)x} \left\langle \{\theta_i\}_{i=1}^k \left|\mathcal{O}\left(0,0\right)\right| \{\vartheta_j\}_{j=1}^l \right\rangle}{\sqrt{\rho_k\left(\{\theta_i\}\right)\rho_l\left(\{\vartheta_j\}\right)}\mathcal{N}_k^*\left(\{\theta_i\}\right)\mathcal{N}_l\left(\{\vartheta_j\}\right)}$$

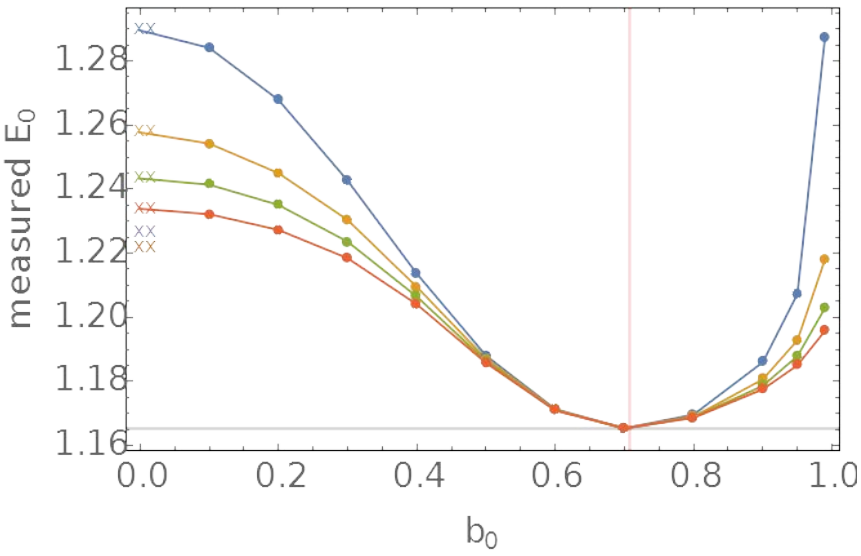
Koubek-Mussardo (1993):

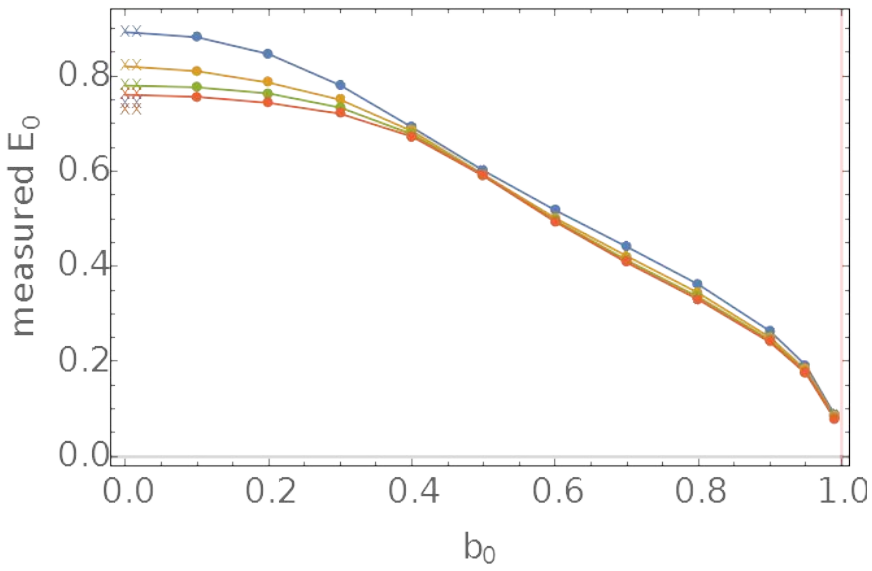
$$\langle 0|e^{kb\phi(0)}|\theta_1,\ldots,\theta_n\rangle \;=\; \langle e^{kb\phi}\rangle\, F_n^{(k)}(\theta_1,\ldots,\theta_n)$$

$$F_n^{(k)}(\theta_1,\ldots,\theta_n) \;=\; H_n^{(k)}\, Q_n^{(k)}(x_1,\ldots,x_n) \prod_{i<j}^{n} \frac{F_{min}(\theta_{ij})}{x_i+x_j}$$

$$Q_n^{(k)}(x_1,\ldots,x_n) \;=\; \det M$$

$$M_{i,j} \propto \sigma_{2i-j}^{(n)}$$





$$b_1 = \frac{1}{\sqrt{2}}$$

$$b_1 = 1$$

When $b_1 = \frac{Q_0}{2}, Q_0 = b_0 + b_0^{-1}$, the perturbation exactly cancels the original potential

For each $b_1 \geq 1$ there exists a b_0 for which this holds
Based on this, Lagrangian sinh-Gordon theory probably describes a massless theory above the self-dual point.

Conclusions

Sinh-Gordon model: simplest interacting QFT in 1+1D

Lagrangian **versus** S-matrix definition

Self-dual S-matrix confirmed for b<1
Presence of **Seiberg bounds** likely

Delicacies with RG improvement: **PT singularities explained**
Supra-Borel resummation
Origins of power-law fit

Form factor TSM: Sanity checks explained
Surprising behavior of cutoff dependence
Hints of massless Lagrangian theory for b>1

Discussion

IS IT SELF-DUAL OR NOT?

Self-dual S-matrix confirmed for b<1

Lagrangian framework may break down for b>1
(Is there a continuum limit of TSM at all?)

Do VEVs with |a|>Q/2 have a continuum limit?
Are they zero?

Do we miss anything (physical) if we restrict to b<=1 and |a|>Q/2?
(Exact S-matrix depends on B, matrix elements periodic in Q)

Conjecture
S-matrix is correct, but B(b) relation only holds for b<1
Lagrangian sinh-Gordon is massless for b>=1 (see also Sklyanin
Operator content is restricted by the Seiberg bounds

Thank you for your attention!

More information:
R. Konik, M. Lájer, G. Mussardo,

Approaching the Self-Dual Point ofthe Sinh-Gordon model,

arXiv:2007.00154 [hep-th]

Accepted for publication at JHEP

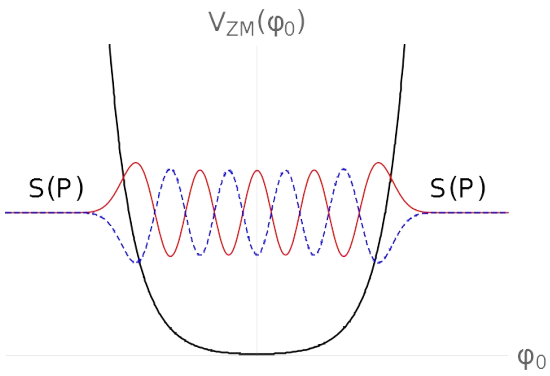
Photo: funiQ

UV behavior

Zero mode quantization

$$H_{\exp} = \frac{2\pi}{L}\left(2\Pi_0^2 + Me^{b\varphi_0}\right) \qquad M = 2\pi\mu_{ShG}\left(\frac{L}{2\pi}\right)^{2+2b^2}$$

$$\psi(x) \simeq e^{iPx} + e^{-iPx}S_{sc}(P) \qquad P = \sqrt{\frac{LE}{4\pi}}$$



Reflection amplitude in exponential potential:

$$S_{sc}(P) = \left(-\left(\frac{L}{2\pi}\right)^{-4iPQ}\left(\frac{\pi\mu_{ShG}}{b^2}\right)^{-\frac{2iP}{b}}\frac{\Gamma(1+2iP/b)}{\Gamma(1-2iP/b)}\right)$$

Approximate quantization in cosh potential:

$$S_{sc}(P)^2 = 1$$

Liouville quantization (UV limit of TBA)

$$E_n = \frac{2\pi}{L}\left(2P_n^2 - \frac{1}{12}\right)$$

$$\rho = \frac{L}{2\pi}\frac{M}{4\sqrt{\pi}}\Gamma\left(\frac{1-B}{2}\right)\Gamma\left(\frac{2+B}{2}\right)$$

$$e^{2i\Theta(P)} = -\bar{b}^{\frac{8iP}{\bar{b}}}\rho^{-4iPQ(\bar{b})}\frac{\Gamma\left(1+2iP\bar{b}\right)\Gamma\left(1+2iP\bar{b}^{-1}\right)}{\Gamma\left(1+2iP\bar{b}\right)\Gamma\left(1+2iP\bar{b}^{-1}\right)}$$

quantization condition:

$$2\,\Theta\left(P\right) = n\pi, \quad n \in \mathbb{Z}_{\geq 0}$$

They are **different**!

$$u = \left[\ln\left(\mu_{ShG}\left(\frac{L}{2\pi}\right)^{2+2b^2}\right)\right]^{-1}$$

Semiclassics:

$$E_0 = \frac{2\pi}{L} b^2 \pi^2 \left(\frac{u^2}{2} - \kappa_S u^3 + \frac{3}{2}\kappa_S^2 u^4 - 2\left(\kappa_S^3 - \frac{\pi^2}{3}\zeta(3)\right) u^5 + \dots\right)$$

TBA (Liouville) reflection:

$$E_0 = \frac{2\pi}{L} b^2 \pi^2 \left(\frac{u^2}{2} - \kappa_L u^3 + \frac{3}{2}\kappa_L^2 u^4 - 2\left(\kappa_L^3 - \frac{\pi^2}{3}\left(1+b^6\right)\zeta(3)\right) u^5 + \dots\right)$$

$$\kappa_S = \left(2\gamma_E + \ln\frac{\pi}{b^2}\right)$$

$$\kappa_L = \left(2\left(1+b^2\right)\gamma_E + \ln\frac{\pi\Gamma\left(b^2\right)}{\Gamma\left(1-b^2\right)}\right)$$

Effective zero-mode potential

$$H^{(\mathrm{ShG})} = H^{(0)}_{\mathrm{cyl}} + \frac{m^2_{\mathrm{eff}}}{16\pi}\int_0^L dx : \tilde{\varphi}^2(x) : + 2\mu_{ShG}\left(\frac{L}{2\pi}\right)^{2b^2}\int_0^L dx : \cosh\left(b\varphi(x,0)\right) : - \frac{m^2_{\mathrm{eff}}}{16\pi}\int_0^L dx : \tilde{\varphi}^2(x) :$$

Allow m_{eff} to depend on φ_0 !
Transcendental equation:

$$m^2_{\mathrm{eff}}(\varphi_0) = 16\pi\mu_{ShG} b^2 \left(\frac{L}{2\pi}\right)^{2b^2} e^{\frac{2\pi}{L} b^2 S_1(m_{\mathrm{eff}}, L)} \cosh b\varphi_0$$

Perform a ***zero-mode dependent*** **Bogoliubov** transform (diagonalizing first two terms of H)

$$H^{\mathrm{eff}}_{ZM} = \left(\frac{4\pi}{L}\Pi_0^2 - \frac{\pi}{6L}\right) + 2\mu_{ShG} L \left(\frac{L}{2\pi}\right)^{2b^2} e^{\frac{2\pi}{L} b^2 S_1(m_{\mathrm{eff}}(\varphi_0), L)} \cosh\left(b\varphi_0\right) + \tilde{S}_2\left(m_{\mathrm{eff}}(\varphi_0), L\right)$$

Small-volume limit from effective potential

Asymptotic Schrödinger equation

$$-y''(x) + re^{\pm x} y(x) - \epsilon^2 e^{\pm 2x} y(x) = c^2 y(x), \quad x \to \pm\infty$$

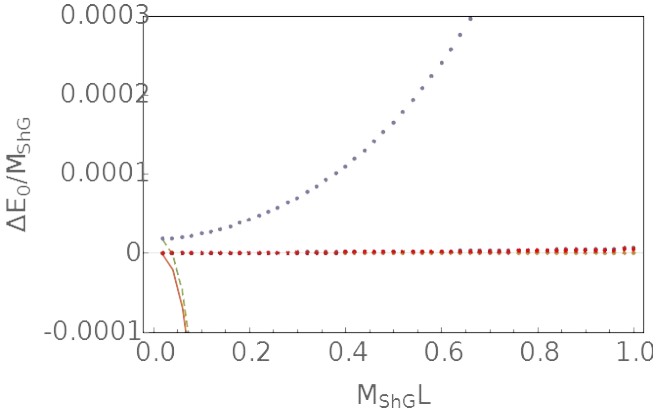
$$x = b\varphi_0 + \ln a, \quad a = \mu_{ShG} L \left(\frac{L}{2\pi}\right)^{2b^2} \qquad r = \frac{L}{4\pi b^2}$$

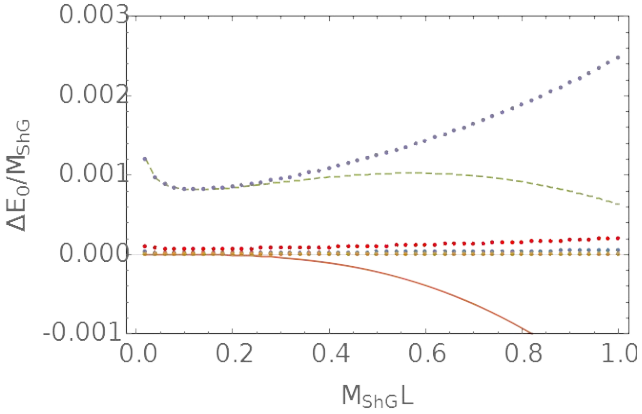
Reflection amplitude

$$S_2(P) = -\left(\frac{L}{2\pi}\right)^{-4iPQ} \left(4\pi i b\, \mu_{ShG} \sqrt{\zeta(3)}\right)^{-\frac{2iP}{b}} \frac{\Gamma\left(1 + \frac{2iP}{b}\right)}{\Gamma\left(1 - \frac{2iP}{b}\right)} \frac{\Gamma\left(\frac{1}{2} - \frac{iP}{b} - \frac{i}{4b^3\zeta(3)}\right)}{\Gamma\left(\frac{1}{2} + \frac{iP}{b} - \frac{i}{4b^3\zeta(3)}\right)}$$

Quantization condition: $$-i \ln\left(-S_2(P)\right) = n\pi$$

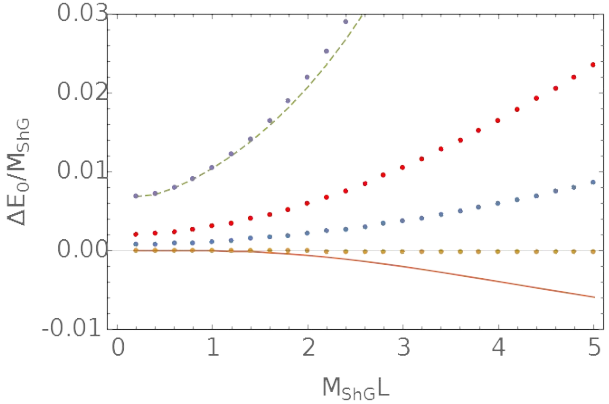
$b = 0.2$

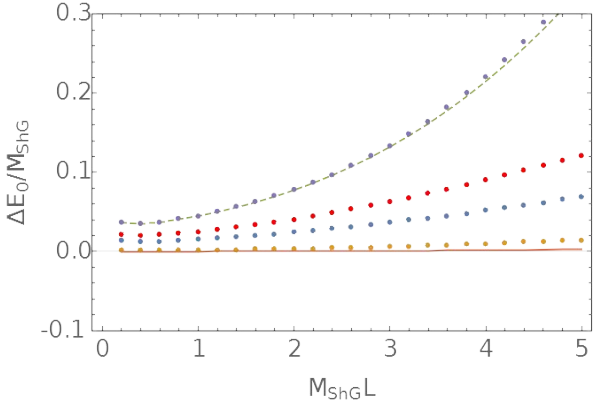




$b = 0.4$

$b = 0.6$





$b = 0.8$

Blue dots: Raw (Nc = 12) TSM data. Orange dots: extrapolated TSM data. Purple dots: eigenvalues of H_{ZM}. Dashed green curve: semi-classical reflection quantization. Continuous red curve: Liouville reflection quantization. Red dots: eigenvalues of effective ZM Hamiltonian.