Truncated spectrum methods
and the self-duality of the sinh-Gordon model

Based on Konik, Lájer, Mussardo, arXiv:2007.00154

Lájer Márton Kálmán
(ELTE, Wigner)
(former PhD student of Zoltán Bajnok)

Collaborators:
Robert Konik
Brookhaven National Laboratory CMPMS
Giuseppe Mussardo
SISSA and INFN, Sezione di Trieste

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Introduction
Quantum field theory (QFT)

- Framework for describing the fundamental interactions of Nature: At least 3 out of 4 (weak, EM, strong)
- Effective models in particle, statistical and solid state physics
- In most cases, only approximate solutions available

\[ \mathcal{D}X_i \quad \text{Boundary conditions} \quad S[X_i] = \int d^Dx \mathcal{L}(X_i, \partial \mu X_i) \]

\[ Z[J_i] = \int \mathcal{D}X_i e^{iS[x_i] + J_i X_i} \]

**INPUT**  
- Particle masses
- S matrix
- Operator matrix elements
- n-point functions etc.

**OUTPUT**
QFT = Quantum mechanics + Poincaré invariance

\[ \mathcal{H} \] Hilbert space

Finite speed information propagation

States:
- Ground state (vacuum) \( |\text{vac}\rangle \) \[ E_{\text{vac}} = E_0 \]
- One particle states \( |p^{(i)}\rangle \) \[ E_1 = E_0 + M_i \cosh \vartheta \]
- Two particle states \( |p^{(i)},p^{(j)}\rangle \) \[ E_2 = E_0 + M_i \cosh \vartheta_1 + M_j \cosh \vartheta_2 \]

Time evolution: \[ U(t) = e^{-iHt} \]

\( H \): Continuum spectrum

Measurements: \( D=4 \) (Universe)

Significance of \( D<4 \)
- Experimentsally relevant (e.g. anisotropic materials, trapped quantum gases)
- Toy models (many exact results)
- String theory (Background: \( D=10/11/26 \), but string worldsheet is \( D=2 \))
D=2(=1+1): integrability

Higher spin conserved charges

\[ S_{ij}^{kl}(\vartheta_i - \vartheta_j) \]

P-dependent translation invariance.

- Dynamics completely determined by 2->2 S-matrices
- (Almost) exclusively completely elastic scattering
- At most particle type can change, but not its rapidity

Factorised S-matrix

Finite volume \( L \) (1 particle type):

\[ S(\vartheta_k) = e^{i\delta(\vartheta)} \]

Wave function periodic

\[ j \in \{1..n\} : \quad e^{i p_j L} \prod_{j \neq k} S(\vartheta_j - \vartheta_k) = 1 \]

Discrete spectrum!

Bethe-Yang equation

Polynomial corrections in \( 1/L \)

Excited state TBA momentum quantization

Exact, \( O(e^{-ML}) \) corrections summed up
Energy levels

\[ E(\{I_j\}) = \sum_{j=1}^{n} M \cosh \vartheta_j \]

\[ E(\{I_j\}) = \sum_{j=1}^{n} M \cosh \vartheta_j + \delta E(\{I_j\}) \]

Correlators:

\[ \langle O_1(x)O_2 \rangle = \sum_{n=0}^{\infty} \int \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} \langle \text{vac} |O(x)| \theta_1 \ldots \theta_n \rangle \langle \theta_1 \ldots \theta_n |O(x)| \text{vac} \rangle \]

Form factors

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Exponential</th>
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<tbody>
<tr>
<td>Pozsgay-Takács</td>
<td>Leclair-Mussardo, Pozsgay-Szécsényi-Takács, Negro-Smirnov, Bajnok-Smirnov...</td>
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<td>Bajnok-Wu</td>
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<td>Pozsgay-Takács</td>
<td>Bajnok-Balog-Lájer-Wu, Bajnok-Lájer-Szépfalvi-Vona</td>
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<tr>
<td>Pozsgay-Takács</td>
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Sinh-Gordon model (ShG)

Lagrangian definition (path integral)

\[ \mathcal{L} = \frac{1}{16\pi} (\partial_\nu \varphi)^2 - 2\mu \cosh(b\varphi) \]

integrable

Viewpoints:

\[ \mathcal{L} = \mathcal{L}_{\text{Gauss}} - 2\mu \cosh(b\varphi) \]
sine-Gordon analytic continuation

\[ \mathcal{L} = \mathcal{L}_m - \left( \frac{m^2}{b^2} \cosh(b\varphi) - \frac{m^2}{2} \varphi^2 \right) \]

1 particle, completely elastic scattering

\[ \mathcal{L} = \mathcal{L}_{\text{Liouville}}^b - \mu e^{-b\varphi} = \mathcal{L}_{\text{Liouville}}^{-b} - \mu e^{b\varphi} \]

\[ b \to \frac{1}{b}, \mu \to \tilde{\mu} \]

Duality in analytically continued quantities, Seiberg bounds

S-matrix bootstrap

\[ S(\theta) = \frac{\sinh \theta - i \sin \pi B}{\sinh \theta + i \sin \pi B} \]

Exact finite volume spectrum as function of physical mass

(Relative) form factors (form factor bootstrap)

CONJECTURE:

The above S-matrix describes scattering in the lagrangian theory on the left (after renormalization), furthermore:

\[ B(b) = \frac{b^2}{1 + b^2} \]
Canonical quantization of Lagrangian theory:
Only **one** parameter to renormalize (normal ordering, tadpole=0)

Finite volume Hamiltonian:

\[
H(L) = H_0^{(\alpha)} + Z_\alpha(L) \mu_{ShG} L : \cosh(b\varphi) :_{\alpha,L} \delta_P + \tilde{E}_0^{(\alpha)}(L)
\]

\(Z_\alpha(L), \tilde{E}_0^{(\alpha)}(L)\) depend on the choice of the unperturbed part; exact determination possible if Is a free model

Exact, finite relation between (well-supported conjecture):

\[
\mu_{ShG} \text{ and } M_{ShG}
\]

\[
\mu_{ShG} = \frac{m^2 + 2b^2}{24 + 2b^2} e^{2b^2 \gamma E}
\]

\[
M_{ShG} = \frac{4\sqrt{\pi}}{\Gamma\left(\frac{1}{2+2b^2}\right) \Gamma\left(1 + \frac{b^2}{2+2b^2}\right)} \left[ -\mu_{ShG} \frac{\pi(1 + b^2)}{\Gamma(-b^2)} \right]^{-\frac{1}{2+2b^2}}
\]

b>1 complex!
Similar problems with VEV:

\[ \langle e^{a\varphi} \rangle \sim M_{\text{ShG}}^{-2a^2} \]

\[ \langle e^{a\varphi} \rangle < 0, \quad |\alpha| > \frac{Q}{2}, \quad Q = b + b^{-1} \]

**Step 1:** Expand H(L) to order $b^4$ (time-independent /old-fashioned/ perturbation theory around free massive boson)

- $E_1(L) - E_0(L)$
- (Volume-exact formulae, double sum)
- Contour integral technique
- Lüser-type integrals
- Newly calculated F-term (BBLW)
- Leading Lüser correction agrees with expansion of F-term

Bajnok, Balog, Lájer, Wu (2018)
Spectrum of $H(L)$ is discrete: **Truncated Spectrum Method (TSM)**

\[ H = H_{\text{ZM}} + H_{\text{Osc}} + H_1 \]

1. **Variational method**: introduce an (energy) cutoff $\Lambda$ in the unperturbed Hilbert space; search eigenvectors/eigenvalues in this finite basis
2. Useful to treat quantum mechanical zero mode separately: Particle in a box of width $D$ with cosh potential
3. Try to correct effect of cutoff $\Lambda$
4. Dependence on $\Lambda$ negligible for small $b$

- BY equations -> identification of states
- Matrix elements -> numerical finite volume FF
- Confirmation of general F-term

Bajnok, Lájer, Szépfalvi, Vona (2019)
1. Cutoff dependence
(Massless) truncated TSM basis

\[ \mathcal{H} = \left\{ a_{n_1}^{\dagger} \cdots a_{n_k}^{\dagger} |0\rangle \otimes a_{-m_1}^{\dagger} \cdots a_{-m_l}^{\dagger} |0\rangle \otimes |s\rangle \sum_{i=1}^{k} n_k \leq N_c \sum_{m=1}^{l} n_l \leq N_c; s = 1, \cdots, N_{ZM} \right\} \]

\[ H_0 = H_{ZM} + H_{NZM} \]

\[ H_{ZM} = \frac{4\pi}{L} \Pi_0^2 + \mu_{ShG} L \left( \frac{L}{2\pi} \right)^2 b^2 \left[ e^{b\varphi_0} + e^{-b\varphi_0} \right] \]

\[ H_{NZM} = \frac{2\pi}{L} \left( L_0 + \bar{L}_0 - \frac{1}{12} \right) \quad L_0 = \sum_{n>0} n a_n^{\dagger} a_n \quad \bar{L}_0 = \sum_{n<0} |n| a_n^{\dagger} a_n \]
Perturbation

\[ \int_0^L dx \, V_{pert}(x) = \delta_{P,0} \mu S h G \left( \frac{L}{2\pi} \right)^2 L \left[ e^{b\varphi_0} \left( : e^{b\tilde{\varphi}(0)} : - 1 \right) + e^{-b\varphi_0} \left( : e^{-b\tilde{\varphi}(0)} : - 1 \right) \right] \]

\[ \varphi(x, \tau) = \varphi_0(\tau) + \tilde{\varphi}(x, \tau) \equiv \varphi_0(\tau) + \varphi_R(x, \tau) + \varphi_L(x, \tau) + \varphi_R^\dagger(x, \tau) + \varphi_L^\dagger(x, \tau) \]

Normal ordering:

\[ : e^{b\varphi(x, \tau)} : \equiv e^{b\varphi_0(\tau)} e^{b\varphi_R^\dagger(x, \tau)} e^{b\varphi_R(x, \tau)} e^{b\varphi_L^\dagger(x, \tau)} e^{b\varphi_L(x, \tau)} \]

\[ \varphi_R(x, \tau) = \sum_{n>0} \sqrt{\frac{2}{|n|}} a_n e^{i k_n x - k_n \tau} \quad \varphi_L(x, \tau) = \sum_{n<0} \sqrt{\frac{2}{|n|}} a_n e^{i k_n x - |k_n| \tau} \]
Finite volume ($M_{\text{ShG}}L=1$) energy levels
Finite volume \( (M_{\text{ShG}} L=1) \) energy levels
Accuracy with respect to TBA
Mass and vacuum energy density
S matrix parameter B

$$S(\theta) = e^{i\delta(\theta)}$$

BY equation:

$$\delta(2\theta) + M_{ShG} L \sinh(\theta) = 2\pi n$$

Two-particle state:

$$\theta = \text{Arcosh} \left( \frac{E - E_0}{2M_{ShG}} \right)$$
Vacuum expectation values

\[ \langle e^{\alpha \phi} \rangle \equiv M_{S h G}^{-2\alpha^2} G(\alpha) \]

\[ b = \frac{2}{\sqrt{8\pi}} \approx 0.4 \]

\[ b = \frac{4}{\sqrt{8\pi}} \approx 0.8 \]

\[ \mu_{S h G} = 0.1, \; L = 6 \]
Vacuum expectation values

\[ \langle e^{\alpha\phi} \rangle \equiv M_{ShG}^{-2\alpha^2} G(\alpha) \]

\[ b = \frac{4}{\sqrt{8\pi}} \approx 0.8 \]

\[ b = \frac{5}{\sqrt{8\pi}} \approx 0.99 \]

\[ \mu_{ShG} = 0.1, L = 2 \]
Increasing $b$: $\Lambda$-dependence strengthens

- Techniques exist to decrease cutoff dependence (Pl. Lencsés-Takács, Hogervorst-Rychkov-van Rees, Elias-Miró-Rychkov-Vitale,...)
- All based on an expansion in a coupling constant
- Expansion in $b$: OK, but we get a Landau-Ginzburg model
- Expansion in $\mu_{\text{ShG}}$:

$$H_{\text{eff}} = H_{ll} + \delta H$$

$$\delta H = H_{lH} \frac{1}{E_* - H_{HH}} H_{Hl}$$

$$\delta H = \sum_{n=2}^{\infty} \delta H_n \mu_{\text{ShG}}^n$$

$$\delta H_2 \propto \Lambda^4 b^2 - 2$$

$\text{Singular for}\quad b \geq \frac{1}{\sqrt{2}}$

$$\delta H_n \propto \Lambda^{2(n^2-n)b^2-2n+2}$$

$\text{Singular for}\quad b \geq \frac{1}{\sqrt{n}}$

$\text{All } b > 0: \text{infinitely many singular terms!}$

TSM numerics show no trace of any of these divergences!
Toy example

\[ f(\mu, \epsilon) = \int_{\epsilon}^{1} dx (x + \mu)^{\frac{1}{b^2} - 1} \]

\[ f(\mu, 0) = b^2 \left( (1 + \mu)^{\frac{1}{b^2}} - \mu^{\frac{1}{b^2}} \right) \]

Attempt an expansion around \( \mu = 0 \):

\[ f(\mu, \epsilon) = \sum_{k=0}^{\infty} \mu^k \binom{b^{-2} - 1}{k} \int_{\epsilon}^{1} x^{b^{-2} - 1 - k} = \sum_{k=0}^{\infty} \mu^k \binom{b^{-2} - 1}{k} \frac{1}{k - b^{-2}} (\epsilon^{b^{-2} - k - 1}) \]

Singular as \( \epsilon \to 0 \) for \( k > b^{-2} \)
Observation 1: Same type of divergences appear in expansion of energy levels

True $\mu$-dependence of energy levels from small volume expansion of TBA:

$$E_n = \frac{1}{L} \sum_{k} \frac{\alpha_k}{\log^k \mu} + O(\mu)$$

Observation 2: TSM data can be fitted robustly with the function $E(\Lambda) = E_{\text{extrap}} + c\Lambda^\gamma$

New strategy: Try to resum the series $E(\Lambda) = \sum_n \alpha_n \Lambda^{2(n^2-n)b^2-2n+2}$ for finite $\Lambda$ and study the asymptotics of the resummed function. It can be shown (with reasonably weak assumptions), that the asymptotics is indeed a power-law, and the predicted exponent is

$$\gamma = \frac{(b^2 - 1)^2}{2b^2}$$
Extra terms to the Hamiltonian? (e.g. $\tilde{\mu}e^{1 \over b^4}$)

- Idea: examine the UV spectrum
- Most precise results of TSM expected at small volumes
- $L \to 0$: Energy of oscillator states diverges, naively one expects the zero mode to dominate
- Instead, the differences between zero-mode and exact energies diverge at $L \to 0$

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<th>analytic</th>
<th>numeric</th>
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<tr>
<td>Semi-classical(ZM)</td>
<td>$S_0^2(P) = 1$</td>
<td>$S_L^2(P) = 1$</td>
</tr>
<tr>
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<td>$O(b^8)$ error</td>
<td>$O(b^{12})$ error</td>
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Both analytical and numerical confirmation that the presence of oscillators completely explains the $L \to 0$ behavior. By all accounts there’s no need of any extra terms in the (bare) potential.
2. Form factor TSM
Form factor TSM

Based on what has been learned so far: construct the Hamiltonian on the basis of eigenstates of a different sinh-Gordon model (using form factors)

\[ H = H_{b_0}^{(ShG)} + H_1 \]

\[ H_{b_0}^{(ShG)} = H_0 + 2\mu_0 \int_0^L dx \cosh b_0 \varphi \]

\[ H_1 = \int_0^L dx \left( 2\mu_1 \cosh b_1 \varphi - 2\mu_0 \cosh b_0 \varphi \right) \]

Sanity checks:
- Reproducing matrix elements of massive basis
- Exact mass-coupling relation from form factor perturbation theory

\[ b_0 \to 0 \]

\[ b_1 = b_0 + \epsilon \]
Double truncation: number of particles $N_{\text{max}}$, momentum (quantization number) cutoff $k_{\text{max}}$

Basis size: up to 1174 dimensions (each parity sector)

Volume range: $5 \leq M_{\text{ShG}} L \leq 15$

Exponential corrections neglected:

➔ Matrix elements of $H_0$ from Bethe-Yang energies
➔ Matrix elements of $H_1$: Koubek-Mussardo, Lukyanov-Zamolodchikov, Pozsgay-Takács formulae
Matrix elements

Pozsgay-Takács (2007), non-diagonal case:

\[
\left\langle \{I_i\}_{i=1}^{k} | \mathcal{O}(x, 0) | \{\tilde{I}_j\}_{j=1}^{l} \right\rangle_L = \frac{e^{iM \left( \sum_{j=1}^{l} \sinh \varphi_j - \sum_{i=1}^{k} \sinh \theta_i \right) x} \left\langle \{\theta_i\}_{i=1}^{k} | \mathcal{O}(0, 0) | \{\varphi_j\}_{j=1}^{l} \right\rangle}{\sqrt{\rho_k(\{\theta_i\}) \rho_l(\{\varphi_j\}) N^*_k(\{\theta_i\}) N_l(\{\varphi_j\})}}
\]

Koubek-Mussardo (1993):

\[
\left\langle 0 | e^{k\cdot b \phi(0)} | \theta_1, \ldots, \theta_n \right\rangle = \left\langle e^{k\cdot b \phi} \right\rangle F_n^{(k)}(\theta_1, \ldots, \theta_n)
\]

\[
F_n^{(k)}(\theta_1, \ldots, \theta_n) = H_n^{(k)} Q_n^{(k)}(x_1, \ldots, x_n) \prod_{i<j} \frac{F_{\text{min}}(x_{ij})}{x_i + x_j} \quad Q_n^{(k)}(x_1, \ldots, x_n) = \det M
\]

\[
M_{i,j} \propto \sigma_{2i-j}^{(n)}
\]
When $b_1 = \frac{1}{\sqrt{2}}$, the perturbation exactly cancels the original potential $Q_0 = b_0 + b_0^{-1}$, the perturbation exactly cancels the original potential

For each $b_1 \geq 1$ there exists a $b_0$ for which this holds.

Based on this, Lagrangian sinh-Gordon theory probably describes a massless theory above the self-dual point.
Conclusions

**Sinh-Gordon model**: simplest interacting QFT in 1+1D

Lagrangian *versus* S-matrix definition

**Self-dual S-matrix confirmed for b<1**
Presence of **Seiberg bounds** likely

Delicacies with RG improvement: **PT singularities explained**
Supra-Borel resummation
Origins of power-law fit

Form factor TSM: Sanity checks explained
Surprising behavior of cutoff dependence
**Hints of massless Lagrangian theory for b>1**
Discussion

IS IT SELF-DUAL OR NOT?

Self-dual S-matrix confirmed for $b<1$

Lagrangian framework may break down for $b>1$
(Is there a continuum limit of TSM at all?)

Do VEVs with $|a|>Q/2$ have a continuum limit?
Are they zero?

Do we miss anything (physical) if we restrict to $b<=1$ and $|a|>Q/2$?
(Exact S-matrix depends on $B$, matrix elements periodic in $Q$)

Conjecture

S-matrix is correct, but $B(b)$ relation only holds for $b<1$
Lagrangian sinh-Gordon is massless for $b>=1$ (see also Sklyanin
Operator content is restricted by the Seiberg bounds)
Thank you for your attention!

More information:
R. Konik, M. Lájer, G. Mussardo,

*Approaching the Self-Dual Point of the Sinh-Gordon model*,


Accepted for publication at JHEP

Photo: funiQ
UV behavior
Zero mode quantization

\[ H_{\text{exp}} = \frac{2\pi}{L} \left( 2\Pi_0^2 + Me^{b\varphi_0} \right) \]

\[ M = 2\pi \mu S h G \left( \frac{L}{2\pi} \right)^{2+b^2} \]

\[ \psi(x) \simeq e^{iPx} + e^{-iPx} S_{\text{sc}}(P) \]

\[ P = \sqrt{\frac{LE}{4\pi}} \]

Reflection amplitude in exponential potential:

\[ S_{\text{sc}}(P) = \left( -\left( \frac{L}{2\pi} \right)^{-4iPQ} \left( \frac{\pi \mu S h G}{b^2} \right)^{-\frac{2iP}{b}} \frac{\Gamma(1+2iP/b)}{\Gamma(1-2iP/b)} \right) \]

Approximate quantization in cosh potential:

\[ S_{\text{sc}}(P)^2 = 1 \]
Liouville quantization (UV limit of TBA)

\[ E_n = \frac{2\pi}{L} \left( 2P_n^2 - \frac{1}{12} \right) \]

\[ \rho = \frac{L}{2\pi} \frac{M}{4\sqrt{\pi}} \Gamma \left( \frac{1-B}{2} \right) \Gamma \left( \frac{2+B}{2} \right) \]

\[ e^{2i\Theta(P)} = -\frac{8iP}{\bar{b}} \rho^{-4iPQ(\bar{b})} \frac{\Gamma \left( 1 + 2iP\bar{b} \right) \Gamma \left( 1 + 2iP\bar{b}^{-1} \right)}{\Gamma \left( 1 + 2iP\bar{b} \right) \Gamma \left( 1 + 2iP\bar{b}^{-1} \right)} \]

quantization condition:

\[ 2\Theta(P) = n\pi, \quad n \in \mathbb{Z}_{\geq 0} \]
They are different!

Semiclassics:

\[ E_0 = \frac{2\pi}{L} b^2 \pi^2 \left( \frac{u^2}{2} - \kappa_S u^3 + \frac{3}{2} \kappa_S^2 u^4 - 2 \left( \kappa_S^3 - \frac{\pi^2}{3} \zeta(3) \right) u^5 + \ldots \right) \]

TBA (Liouville) reflection:

\[ E_0 = \frac{2\pi}{L} b^2 \pi^2 \left( \frac{u^2}{2} - \kappa_L u^3 + \frac{3}{2} \kappa_L^2 u^4 - 2 \left( \kappa_L^3 - \frac{\pi^2}{3} \left(1 + b^6\right) \zeta(3) \right) u^5 + \ldots \right) \]

\[ \kappa_S = \left( 2\gamma_E + \ln \frac{\pi}{b^2} \right) \]

\[ \kappa_L = \left( 2 \left(1 + b^2\right) \gamma_E + \ln \frac{\pi \Gamma \left(b^2\right)}{\Gamma \left(1 - b^2\right)} \right) \]
Effective zero-mode potential

\[ H^{(\text{ShG})} = H^{(0)}_{\text{cyl}} + \frac{m_{\text{eff}}^2}{16\pi} \int_0^L dx : \bar{\varphi}^2 (x) : + 2\mu_{\text{ShG}} \left( \frac{L}{2\pi} \right)^2 \int_0^L dx : \cosh (b \varphi (x, 0)) : - \frac{m_{\text{eff}}^2}{16\pi} \int_0^L dx : \bar{\varphi}^2 (x) : \]

Allow \( m_{\text{eff}} \) to depend on \( \varphi_0 \)!

Transcendental equation:

\[ m_{\text{eff}}^2 (\varphi_0) = 16\pi \mu_{\text{ShG}} b^2 \left( \frac{L}{2\pi} \right)^2 e^{\frac{2\pi b^2 S_1 (m_{\text{eff}}, L)}{L}} \cosh b\varphi_0 \]

Perform a zero-mode dependent Bogoliubov transform (diagonalizing first two terms of \( H \))

\[ H^{\text{eff}}_{\text{ZM}} = \left( \frac{4\pi}{L} \Pi_0^2 - \frac{\pi}{6L} \right) + 2\mu_{\text{ShG}} L \left( \frac{L}{2\pi} \right)^2 e^{\frac{2\pi b^2 S_1 (m_{\text{eff}} (\varphi_0), L)}{L}} \cosh (b\varphi_0) + \tilde{S}_2 \left( m_{\text{eff}} (\varphi_0), L \right) \]
Small-volume limit from effective potential

Asymptotic Schrödinger equation

\[-y''(x) + re^{\pm x}y(x) - \epsilon^2 e^{\pm 2x}y(x) = c^2 y(x), \quad x \to \pm \infty\]

\[x = b\varphi_0 + \ln a, \quad a = \mu_{ShG}L \left( \frac{L}{2\pi} \right)^{2b^2} \quad r = \frac{L}{4\pi b^2}\]

Reflection amplitude

\[S_2(P) = -\left( \frac{L}{2\pi} \right)^{-4iPQ} \left( 4\pi i b \mu_{ShG} \sqrt{\zeta(3)} \right)^{-\frac{2iP}{b}} \frac{\Gamma(1 + \frac{2iP}{b}) \Gamma \left( \frac{1}{2} - \frac{iP}{b} - \frac{i}{4b^3 \zeta(3)} \right)}{\Gamma \left( 1 - \frac{2iP}{b} \right) \Gamma \left( \frac{1}{2} + \frac{iP}{b} - \frac{i}{4b^3 \zeta(3)} \right)}\]

Quantization condition:

\[-i \ln (-S_2(P)) = n\pi\]