Lüscher corrections for non-diagonal form factors in integrable QFTs

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Outline

Introduction to integrability

Form factors

Finite volume physics

Nondiagonal 1-particle form factor

Conclusions

Recent Work

- [arXiv:1802.04021] Z. Bajnok, J. Balog, M. Lájer, C. Wu, "Field theoretical derivation of Lüscher's formula and calculation of finite volume form factors", (2018)

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Integrable QFT

- Defining property: ∞ number of (higher spin) coserved charges
- Examples:
  - Free theories

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{2}\phi^2$$

$$\mathscr{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi$$

  - some 1+1D theories with nontrivial scattering:

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{b^2}\left(\cosh b\phi - 1\right) \quad \text{sinh-Gordon}$$

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{\beta^2}\left(\cosh \beta\phi - 1\right) \quad \text{sine-Gordon}$$

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{6g^2}\left(2e^{g\phi} + e^{-2g\phi}\right) \quad \text{Bullough-Dodd}$$

$$\mathscr{L} = \frac{1}{8\pi}\left(\partial_\mu \phi\right)\cdot\left(\partial^\mu \phi\right) - \frac{\mu^2}{16\lambda^2}\sum_{i=1}^{r+1} q_i\left(e^{\beta\alpha_i\cdot\phi} - 1\right) \quad \text{Toda hierarchy}$$

Conserved charges (cl. sinh-Gordon)

- Lightcone coordinates:

$$x_\pm = \frac{1}{2}(t \pm x)$$

- sinh-Gordon EOM $(m = 1, \phi \to b\phi)$

$$\partial_+ \partial_- \phi = -\sinh \phi$$

- Conserved currents:

$$\partial_\mu J^{(s)\mu} = 0 \qquad \Longleftrightarrow \partial_+ J_-^{(s)} + \partial_- J_+^{(s)}$$

- Conserved charges:

$$Q^{(s)} = \int_{-\infty}^{\infty} dx J_0^{(s)}$$

Conserved charges (cl. sinh-Gordon)

- Bäcklund transformation: Let ϕ be a solution of the EOM. Then $\hat{\phi}$ also a solution if

$$\partial_- \left(\phi - \hat{\phi}\right) = 2\varepsilon \sinh\left[\frac{1}{2}\left(\phi + \hat{\phi}\right)\right]$$

$$\partial_+ \left(\phi + \hat{\phi}\right) = \frac{2}{\varepsilon} \sinh\left[\frac{1}{2}\left(\hat{\phi} - \phi\right)\right]$$

- Expand $\hat{\phi}\left(x_+, x_-, \varepsilon\right)$ in ε and solve order by order

$$\hat{\phi} \equiv \hat{\phi}\left(x_+, x_-, \varepsilon\right) = \sum_{n=0}^{\infty} \hat{\phi}_n\left(x_+, x_-\right)\varepsilon^n$$

$$\hat{\phi}_0 = 0 \qquad \hat{\phi}_1 = 2\partial_+\phi$$

$$\hat{\phi}_2 = 2\partial_+^2\phi \qquad \hat{\phi}_3 = 2\partial_+^3\phi - \frac{1}{3}\left(\partial_+\phi\right)^3$$

$$\hat{\phi}_4 = 2\partial_+^4\phi - 2\left(\partial_+\phi\right)^2\partial_+^2\phi$$

Conserved charges (cl. sinh-Gordon)

- An useful identity (for ϕ solving the EOM):

$$\partial_- \left(\frac{1}{2} [\partial_+ \phi]^2 \right) + \partial_+ (\cosh \phi - 1) = 0$$

- Substitute series expansion for $\hat{\phi}$: conserved current for each power of ε

$$J_+^{(1)} \equiv T_2 = \frac{1}{2} (\partial_+ \phi)^2$$

$$J_+^{(3)} \equiv T_4 = 2 \left(\partial_+^2 \phi \right)^2 + 2 \partial_+ \phi \partial_+^3 \phi$$

$$J_+^{(5)} \equiv T_6 = 2 \left(\partial_+^3 \phi \right)^2 + 4 \left(\partial_+^2 \phi \right) \partial_+^4 \phi - 6 \left(\partial_+^2 \phi \right)^2 (\partial_+ \phi)^2 - 2 (\partial_+ \phi)^3 \partial_+^3 \phi + 2 \partial_+ \phi \partial_+^5 \phi$$

- At quantum level:

$$\left[Q^{(s)}, Q^{(s')} \right] = 0$$

S-matrix generalities

- S-matrix definition:

$$S_{fi} = \langle f, out \mid i, in \rangle = \langle f, in | S | i, in \rangle$$

$$|out\rangle = S^{-1} |in\rangle$$

- Unitarity: $|\psi\rangle = c_n |n\rangle$, $\sum_n |c_n|^2 = 1$

$$1 = \sum_m |\langle m | S | \psi \rangle|^2 = \sum_{n_1 n_2} c_{n_1}^* c_{n_2} \left\langle n_1 \middle| S^\dagger S \middle| n_2 \right\rangle \quad \forall c_{n_1}$$

$$\Longrightarrow S^\dagger S = \mathbf{1}$$

- Separation of interaction part:

$$S_{fi} = \delta_{fi} + i (2\pi)^d \delta^d (p_f - p_i) T_{fi}$$

- Implication of unitarity (optical theorem):

$$T_{fi} - T_{if}^* = i (2\pi)^d \sum_{|n\rangle \in \mathscr{H}} \delta^d (p_n - p_i) T_{fn} T_{in}^*$$

$2 \to 2$ S-matrix elements

- $2 \to 2$ scattering matrix elements: $|i\rangle = |p_1, p_2\rangle$, $|f\rangle = |p_3, p_4\rangle$

$$\left\langle p_3, p_4 \left| \frac{1}{i}\left(T - T^\dagger\right) \right| p_1, p_2 \right\rangle = (2\pi)^d \sum_{|n\rangle \in \mathscr{H}} \delta^d\left(p_n - p_i\right) \cdot$$
$$\langle p_3, p_4 \,|\, T \,|\, n \rangle \left\langle n \left| T^\dagger \right| p_1, p_2 \right\rangle$$

- $d = 2$: separate a δ^2 from the S-matrix element

$$\langle p_3, p_4 \,|S|\, p_1, p_2 \rangle \propto (2\pi)^2 \, \delta^2\left(p_1 + p_2 - p_3 - p_4\right) S\left(s, t, u\right)$$
$$s = \left(p_1 + p_2\right)^2, \quad t = \left(p_1 - p_3\right)^2, \quad u = \left(p_1 - p_4\right)^2$$

- Purely elastic scattering: $p_1 = p_4$, $S\left(s, t, u\right) \equiv S\left(s, t\right)$
- Mandelstam variable identity: $s + t + u = \sum_{i=1}^4 m_i^2$, so $S\left(s, t\right) \equiv S\left(s\right)$
- Crossing symmetry:

$$S\left(s + i\varepsilon\right) = S\left(t + i\varepsilon\right) = S\left(2m_1^2 + 2m_2^2 - s - i\varepsilon\right)$$

- Time reversal:

$$S\left(s + i\varepsilon\right) = S\left(s - i\varepsilon\right)^{-1}$$

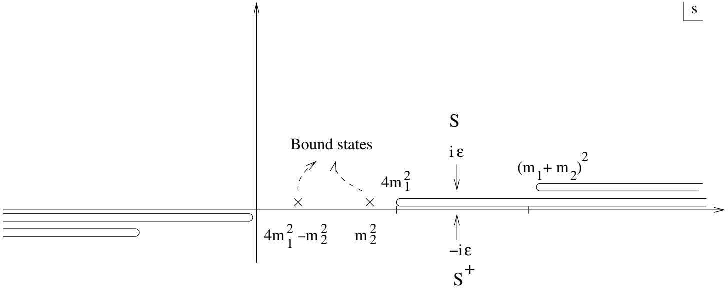
2→ 2 elastic S-matrix analytic structure

- Implication of optical theorem for $S(s)$:

$$2\Re e S(s) = \int dk_n \delta^2(p_1 + p_2 - p_n)\Gamma^n_{12}\Gamma^{34}_n +$$

$$\int dk_{n_1} dk_{n_2} \delta^2(p_1 + p_2 - p_{n_1} - p_{n_2}) S(\{p_1, p_2, p_{n_1}, p_{n_2}\}) S(\{p_{n_1}, p_{n_2}, p_3, p_4\}) +$$

$$\int dk_{n_1} dk_{n_2} dk_{n_3} \ldots$$



Rapidity, states

- Efficient paramtetrization of particle momenta:

$$p_i^{(0)} = m_i \cosh \theta_i \quad p_i^{(1)} = m_i \sinh \theta_i$$

- Action of Lorentz transform: $\theta \to \theta + \alpha$
- n-particle state:

$$|A_{a_1}(\theta_1) A_{a_2}(\theta_2) \dots A_{a_n}(\theta_n)\rangle$$

- in state ordering: $\theta_1 > \theta_2 > \cdots > \theta_n$
- out state ordering: $\theta_1 < \theta_2 < \cdots < \theta_n$
- customary normalization:

$$\langle A_i(\theta_1) \mid A_j(\theta_2) \rangle = 2\pi \delta_{ij} \delta(\theta_1 - \theta_2)$$

Effect of higher spin charges

- Conserved charges:

$$Q_1 = P = P^{(0)} + P^{(1)}$$
$$Q_{-1} = \bar{P} = P^{(0)} - P^{(1)}$$

- All charges commute

$$[Q_s, Q_{s'}] = 0$$

- Common eigensystem:

$$Q_s \left| A_a(\theta) \right\rangle = \omega_s^{(a)} \left| A_a(\theta) \right\rangle$$

- Lorentz transform: $Q_{|s|}$ transforms as s copies of P, $Q_{-|s|}$ as s copies of $\bar{P}$

$$\omega_s^{(a)}(\theta) = \chi_s^{(a)} e^{s\theta}$$

- $\chi_s^{(a)}$ is an attribute of particle type a

Effect of higher spin charges

- Coleman-Mandula theorem (3+1D): higher spin conserved charges $\implies S = \mathbf{1}$
- 1+1D: severe constraints
  - N_a of particles with mass m_a conserved $\forall a$
  - set of final momenta coincide with the set of initial momenta (purely elastic scattering)
  - n-particle scattering completely factorises into $\frac{1}{2}n(n-1)$ two-particle scatterings
- Reason for elasticity:

$$Q_s \, |A_{a_1}(\theta_1) \ldots A_{a_n}(\theta_n)\rangle = \left(\sum_{i=1}^{n} \chi_s^{(a_i)} e^{s\theta_i} \right) |A_{a_1}(\theta_1) \ldots A_{a_n}(\theta_n)\rangle$$

$$\frac{dQ_s}{dt} = 0$$

$$\sum_{i \in |in\rangle} \chi_s^{a_i} e^{s\theta_i} = \sum_{j \in |out\rangle} \chi_s^{a_j} e^{s\theta_j}$$

- Only solution: sets of rapidities equal (permutations among particles with same mass and χ_s^a possible)

Effect of higher spin charges

- (Heuristical) reason of factorisation: use ($\chi_s^{(a)} = 1$ for simplicity)

$$e^{icQ_s} |A_a(p)\rangle = e^{icp^s} |A_a(p)\rangle$$

- Take a localised wavepacket

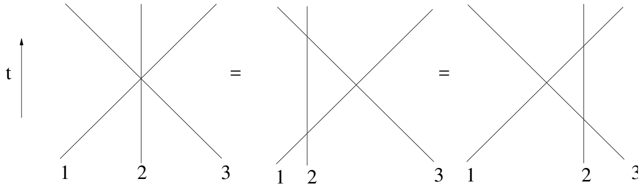
$$\psi(x) = \int_{-\infty}^{\infty} dp e^{-a(p-p_0)^2} e^{ip(x-x_0)}$$

- inspect the effect of symmetry transform generated by Q_s on $\psi(x)$:

$$e^{icQ_s}\psi(x) = \int_{-\infty}^{\infty} dp e^{-a(p-p_0)^2} e^{ip(x-x_0)} e^{icp^s}$$

- Saddle-point approximation $\Longrightarrow$ particle localised at $x' = x_0 - scp^{s-1}$
- momentum-dependent transformation for $s \geq 2$

Effect of higher spin charges



- Yang-Baxter equations

$$S(\{p_1, p_2\})\, S(\{p_2, p_3\})\, S(p_1, p_3) = S(\{p_2, p_3\})\, S(\{p_3, p_1\})\, S(\{p_1, p_2\})$$

- This is necessary and sufficient for the complete factorization of n-particle scattering
- S-matrix theory drastically simplified
- 2-particle S-matrices can be found exactly

S-matrix properties

- 2d S-matrix elements:

$$\left|A_i(\theta_1)A_j(\theta_2)\right\rangle = S_{ij}^{kl}(\theta)\left|A_k(\theta_2)A_l(\theta_1)\right\rangle, \quad \theta_1 > \theta_2$$

$$\theta \equiv \theta_1 - \theta_2$$

- Different normalization of states $\Longrightarrow$ different normalization of S-matrix elements

$$(\text{earlier})\, S_{ij}^{kl}(s) = 4m_i m_j \sinh\theta\, S_{ij}^{kl}(\theta)$$

- Mandelstam variables ($\theta_{ij} \equiv \theta_i - \theta_j$):

$$s(\theta_{ij}) = (p_i + p_j)^2 = m_i^2 + m_j^2 + 2m_i m_j \cosh\theta_{ij}$$
$$t(\theta_{ij}) = (p_i - p_j)^2 = s(i\pi - \theta_{ij})$$

- $S_{ij}^{kl}(\theta)$: r types of particles $\Longrightarrow$ r^4 functions

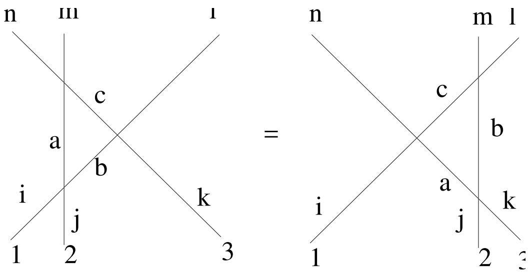
S-matrix properties

- Discrete symmetries

$$P: \quad S_{ij}^{kl} = S_{ji}^{lk}$$
$$C: \quad S_{ij}^{kl} = S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}$$
$$T: \quad S_{ij}^{kl} = S_{lk}^{ji}$$

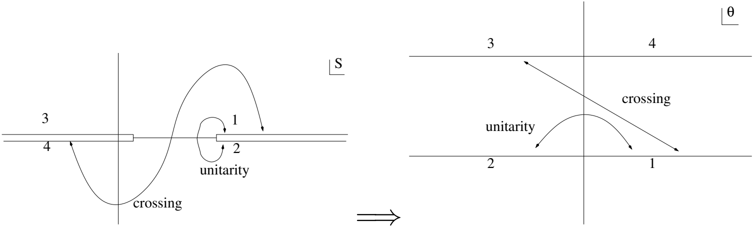
- Yang-Baxter equations:

$$S_{ij}^{ab}(\theta_{12})\, S_{bk}^{cd}(\theta_{13})\, S_{ac}^{nm}(\theta_{23}) = S_{jk}^{ab}(\theta_{23})\, S_{ia}^{nc}(\theta_{13})\, S_{cb}^{ml}(\theta_{12})$$



S-matrix properties

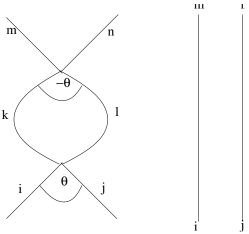
- Analytic properties:
  - only 1 pair of branch cuts: $s \geq (m_i + m_j)^2$, $s \leq (m_i - m_j)^2$ (no particle production)
  - S is a real analytic function $S_{ij}^{kl}(s^*) = \left[S_{ij}^{kl}(s)\right]^*$
- $S_{ij}^{kl}(s) \to S_{ij}^{kl}(\theta)$: $s(\theta_{ij}) = m_i^2 + m_j^2 + 2 m_i m_j \cosh \theta_{ij}$



S-matrix properties

- Unitarity $(s^+ \equiv s + i0,\ s > (m_i + m_j)^2$

$$S_{ij}^{kl}(s^+)\left[S_{kl}^{mn}(s^+)\right]^* = \delta_i^m \delta_j^n \iff S_{ij}^{kl}(\theta)\, S_{kl}^{mn}(-\theta) = \delta_i^m \delta_j^n$$



- Crossing

$$S_{ij}^{kl}(s^+) = S_{i\bar{l}}^{k\bar{j}}\left(2m_i^2 - 2m_j^2 - s^+\right) \iff S_{ij}^{kl}(\theta) = S_{i\bar{l}}^{k\bar{j}}(i\pi - \theta)$$

Bound states

- S-matrix poles

$$S_{ij}^{kl}(\theta) \simeq i\frac{R^{(n)}}{\theta - iu_{ij}^{n}}$$

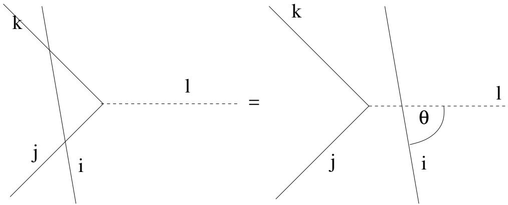
$$s\left(u_{ij}^{n}\right) \equiv m_n^2 = m_i^2 + m_j^2 + 2m_i m_j \cos u_{ij}^{n}$$

- Particles appear as bound states of each other

$$u_{ij}^{n} + u_{in}^{j} + u_{jn}^{i} = 2\pi$$

- S-matrix bootstrap (diagonal scattering for simplicity):

$$S_{il}(\theta) = S_{ij}\left(\theta + i\bar{u}_{jl}^{k}\right) S_{ik}\left(\theta - i\bar{u}_{lk}^{j}\right), \quad \bar{u}_{ab}^{c} = \pi - u_{ab}^{c}$$



Exact S-matrices

- Building block:

$$f_x(\theta) = \frac{\sinh\theta + i\sin\pi x}{\sinh\theta - i\sin\pi x}$$

- Sinh-Gordon model ($r = 1$, no bound state)

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{b^2}(\cosh b\phi - 1)$$

$$S(\theta) = f_{-B}(\theta), \quad B(b) = \frac{b^2}{8\pi + b^2}$$

- Bullough-Dodd model (r=1, self bound state)

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{6g^2}\left(2e^{g\phi} + e^{-2g\phi}\right)$$

$$S(\theta) = f_{\frac{2}{3}}(\theta)\, f_{-\frac{G}{3}}(\theta)\, f_{\frac{G-2}{3}}(\theta), \quad G(g) = \frac{2g^2}{4\pi + g^2}$$

Exact S-matrices

- sine-Gordon model: complicated spectrum (kink K, antikink $\bar{K}$, breathers B_n)

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{\beta^2}\left(\cos\beta\phi - 1\right)$$

$$\xi = \frac{\beta^2}{8\pi - \beta^2}$$

- Kink/antikink scattering nondiagonal

$$S^{SG}(\theta) = \begin{pmatrix} S & & & \\ & S_T & S_R & \\ & S_R & S_T & \\ & & & S \end{pmatrix}, \quad S(\theta) = -\exp\left[-i\int_0^t \frac{dt}{t}\frac{\sinh\frac{t(\xi-\theta)}{2}}{\sinh\frac{\xi t}{2}\cosh\frac{\pi t}{2}}\sin\theta t\right]$$

$$S_T(\theta) = \frac{\sinh\frac{\pi\theta}{\xi}}{\sinh\frac{\pi(i\pi-\theta)}{\xi}}S(\theta) \quad S_R(\theta) = i\frac{\sin\frac{\pi^2}{\xi}}{\sinh\frac{\pi(i\pi-\theta)}{\xi}}S(\theta)$$

- Number of breathers: $N = \left\lfloor \frac{\pi}{\xi} \right\rfloor$, KB_n, $\bar{K}B_n$, B_nB_m S-matrices also known exactly

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Motivation

- How to calculate correlation functions (of local, scalar operators)?
- In 2D a good strategy is to insert complete systems (of in/out states)

$$\mathbf{1} = \sum_{n=0}^{\infty} \int \frac{d\theta_1 \dots d\theta_n}{n!\,(2\pi)^n} \left|\theta_1,\dots,\theta_n\right\rangle \left\langle \theta_1,\dots,\theta_n\right|$$

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle = \sum_{n=0}^{\infty} \frac{d\theta_1 \dots d\theta_n}{n!\,(2\pi)^n} \langle 0|\mathcal{O}(x)|\,\theta_1 \dots \theta_n\rangle_{in} \langle \theta_1 \dots \theta_n|\mathcal{O}(0)|0\rangle$$

- This series has very good convergence properties.
- A form factor is a matrix element between an in and an out state

$$\langle \theta_{m+1} \dots \theta_n|\mathcal{O}(x)|\,\theta_1 \dots \theta_m\rangle = \langle 0|\mathcal{O}(x)|\,\theta_1 \dots \theta_m, \theta_{m+1} - i\pi, \dots \theta_n - i\pi\rangle \\ + \text{disc.}$$

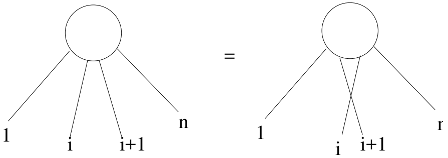
- The elementary form factor $F_n^{\mathcal{O}}$ is defined as

$$F_n^{\mathcal{O}}(\theta_1 \dots \theta_n) := \langle 0|\mathcal{O}(0)|\,\theta_1 \dots \theta_n\rangle$$

Form factor axioms

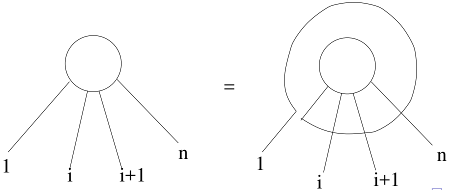
1. Maximal analycity: $F_n^{\mathcal{O}}$ is a meromorphic function, and its poles have physical origin
2. permutation property

$$F_n(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n) = S(\theta_i-\theta_{i+1})\,F_n(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n)$$



3. periodicity property

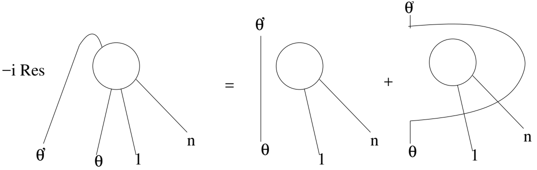
$$F_n(\theta_1+2\pi i,\theta_2\ldots\theta_n) = F_n(\theta_2,\ldots,\theta_n,\theta_1) = \prod_{i=2}^{n} S\left(\theta_i-\theta_j\right) F_n(\theta_1,\ldots,\theta_n)$$



Form factor axioms

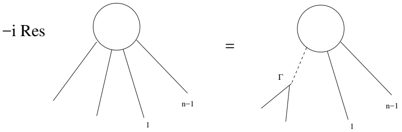
4. kinematical singularity

$$-i \lim_{\theta' \to \theta} \left(\theta' - \theta\right) F_{n+2}\left(\theta' + i\pi, \theta, \theta_1, \ldots, \theta_n\right) = \left(1 - \prod_{i=1}^{n} S\left(\theta - \theta_i\right)\right) F_n\left(\theta_1 \ldots \theta_n\right)$$



5. dynamical singularity: if the S-matrix has a simple pole
$-i \lim_{\theta \to i u_{ij}^k} \left(\theta - i u_{ij}^k\right) S_{ij}(\theta) = \left(\Gamma_{ij}^k\right)^2$, then

$$-i \lim_{\varepsilon \to 0} F_{n+1}\left(\theta + i\bar{u}_{ik}^j - \varepsilon, \theta - i\bar{u}_{jk}^i + \varepsilon, \theta_1 \ldots \theta_{n-1}\right) = \Gamma_{ij}^k F_n\left(\theta, \theta_1, \ldots \theta_{n-1}\right)$$



Form factor bootstrap

- Axioms 1-3 satisfied by taking $F_n^{\mathscr{O}}$ in the following form

$$F_n^{\mathscr{O}} = K_n^{\mathscr{O}}(\theta_1, \ldots \theta_n) \prod_{i<j} F_{min}(\theta_{ij})$$

- where $F_{min}(\theta)$ is the minimal 2-particle FF, with the following properties:
  - analytic in $0 \leq \Im\mathrm{m}\theta < \pi$
  - mildest possible behavior as $|\theta| \to \infty$
  - Solution to axioms 2-3 ("Watson equations") with neither zeroes or poles in $0 < \Im\mathrm{m}\theta < \pi$

- The above properties uniquely determine F_{min} up to a normalization factor. Parametrizing the S-matrix as

$$S(\theta) = \exp\left[\int_0^\infty \frac{dt}{t} f(t) \sinh\frac{t\theta}{i\pi}\right]$$

$$F_{min}(\theta) = \mathscr{N} \exp\left[\int_0^\infty \frac{dt}{t} f(t) \sin^2\frac{t\hat{\theta}}{i\pi}\right], \quad \hat{\theta} = i\pi - \theta$$

Form factor bootstrap

- $K_n^{\mathcal{O}}$ factors:
  - solve Watson equations with $S(\theta) = 1$
  - completely symmetric in each variable+period $2\pi i$
  - $\Longrightarrow$ functions of $\cosh\theta_{ij}$
  - contain all physical poles:

$$K_n^{\mathcal{O}}(\theta_1 \dots \theta_n) = \frac{Q_n^{\mathcal{O}}(\theta_1 \dots \theta_n)}{D_n(\theta_1 \dots \theta_n)}$$

  - D_n only depends on the theory, not the operator. All information about $\mathcal{O}$ is contained in $Q_n^{\mathcal{O}}$

sinh-Gordon form factors

Calculation of the minimal 2-particle form factor

$$S(\theta) = \frac{\sinh\theta - i\sin\pi B}{\sinh\theta + i\sin\pi B}$$

$$= \exp\left[8\int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}\left(1-\frac{B}{2}\right)\right)\sinh\frac{x}{2}\sinh\left(\frac{x\theta}{i\pi}\right)}{\sinh x}\right]$$

$$F_{min}(\theta, B) = \mathcal{N}\exp\left[8\int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}\left(1-\frac{B}{2}\right)\right)\sinh\frac{x}{2}}{\sinh^2 x}\sin^2\left(\frac{x\hat{\theta}}{2\pi}\right)\right]$$

choosing the normalization as

$$\mathcal{N} = \exp\left[-4\int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}\left(1-\frac{B}{2}\right)\right)\sinh\frac{x}{2}}{\sinh^2 x}\right]$$

results in

$$\lim_{\theta\to\infty} F_{min}(\theta, B) = 1$$

Since there are only kinematical poles (no bound states), the form factor Ansatz can be written

$$F_n(\theta_1 \dots \theta_n) = H_n Q_n(x_1 \dots x_n) \prod_{i<j} \frac{F_{min}(\theta_{ij})}{x_i + x_j}, \quad x_i \equiv e^{\theta_i}$$

sinh-Gordon form factors

- Since there are only kinematical poles (no bound states), the form factor Ansatz can be written

$$F_n(\theta_1 \dots \theta_n) = H_n Q_n(x_1 \dots x_n) \prod_{i<j} \frac{F_{min}(\theta_{ij})}{x_i + x_j}, \quad x_i \equiv e^{\theta_i}$$

- This has poles exactly at $\theta_i = \theta_j + i\pi$
- Kinematical singularity condition is translated into a recursion on Q_n

$$(-1)^n Q_{n+2}(-x, x, x_1 \dots x_n) = x C_n(x, x_1, \dots, x_n) Q_n(x_1, \dots, x_n)$$

$$C_n = \frac{-i}{4 \sin \frac{\pi B}{2}} \left(\prod_{i=1}^{n} (x + \omega x_i) \left(x - \omega^{-1} x_i \right) - \prod_{i=1}^{n} (x - \omega x_i) \left(x + \omega^{-1} x_i \right) \right)$$

$$\omega = \exp\left(\frac{i\pi B}{2} \right)$$

- Normalization H_n fixed (up to H_1 and H_2) by

$$H_{2n+1} = H_1 \mu^{2n}, \quad H_{2n} = H_2 \mu^{2n-2}, \quad \mu = \left(\frac{4 \sin\left(\frac{\pi B}{2}\right)}{F_{min}(i\pi, B)} \right)^{\frac{1}{2}}$$

sinh-Gordon form factors

- This recursion can be solved!
- A remarkable class of solutions is the "elementary solution" ($k \in \mathbb{Z}$)

$$Q_n(k) = \det M(k)$$

$$M_{ij}(k) = \sigma^{(n)}_{2i-j} \frac{\sin\left[(i-j+k)\frac{B}{2}\right]}{\sin\frac{B}{2}}$$

- where $\sigma^{(n)}_k$ is the completely symmetric polynomial in n variables of total degree k (but linear in each variable).
- Form factors of ϕ, $T_{\mu\nu}$ and $e^{k\phi}$ can be obtained from this solution, corresponding to different choices of H_1, H_2 and k.

Outline

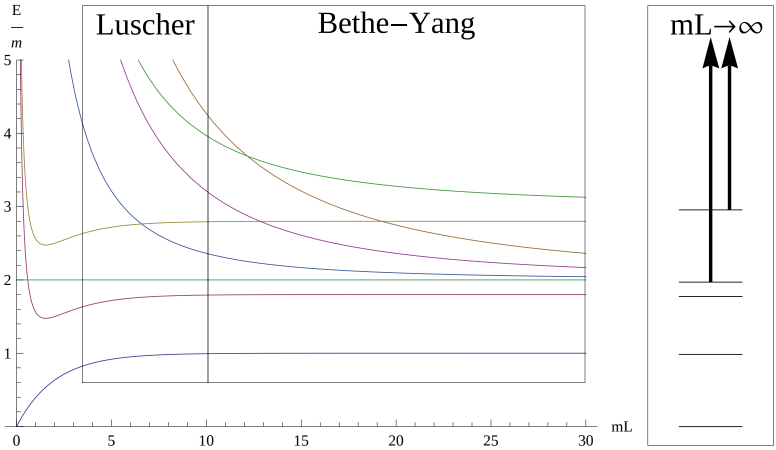
Finite size effects

- "Finite volume": space is compactified via $\varphi(x+L) = \pm\varphi(x)$
- Lüscher:

  finite volume spectrum of $H \Longleftrightarrow$ infinite volume physical quantities

- Leading order: "Bethe-Yang" corrections
  - **Polynomial** in L^{-1}
  - due to momentum quantization (BC)
- "Lüscher" corrections
  - **exponentially** small
  - due to vacuum polarization (virtual particles)

Finite volume spectrum



Polynomial corrections to spectrum

- Momentum quantization (Bethe-Yang):
  - Dimensionless volume large compared to particle number
  - Particles move freely most of the time: $e^{ip_i L}$
  - Particle wave functions pick up phase upon interacting

$$e^{ip_i L} \prod_{j:j\neq i} S_{ij}(\theta_i - \theta_j) = \pm 1, \qquad i = 1 \dots N$$

$$m_i L \sinh \theta_i + \sum_{j\neq i} \delta_{ij}\left(\theta_i - \theta_j\right) = 2\pi n_i, \quad n_i \in \mathbb{Z}$$

  - $\{n_i\}$ determine a state of energy

$$E(L) = \sum_{n=1}^{N} m_n \cosh \theta_n(L)$$

- $\pm$ depends on periodic/antiperiodic nature of wave function

Lüscher corrections to particle mass

- virtual processes (Lüscher)
- Leading order for vacuum:
  - lightest (mass$m$) particle-antiparticle pair appears, travels around the world and annihilate

$$E_0(L) = -m \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \, e^{-mL\cosh\theta}$$

- Leading order for 1-particle state a:
  - **F-term**: virtual pair $b\bar{b}$ from vacuum $\rightarrow$ scattering on $a$$\rightarrow$annihilate:

$$\Delta_F m_a = -m_b \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \, (S_{ba}(i\pi/2+\theta)-1) e^{-m_b L\cosh\theta}$$

  - **μ-term**: residue of F-term at pole bound state c:

$$\Delta_\mu m_a = -\sum_{b,c} \theta(m_a^2 - |m_b^2 - m_c^2|) \mu_{ab}^c (-i) \mathrm{Res} S_{ab}(\theta) e^{-\mu_{ab}^c L}$$

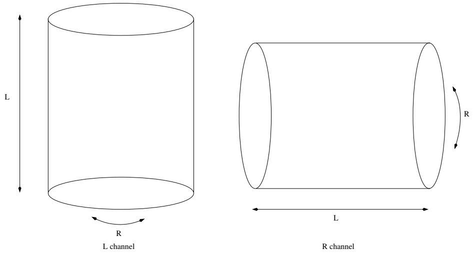
TBA for ground state

- Start from Euclidean field theory on the torus, periodic BC, $L \to \infty$
- 2 equivalent ways of quantization:

$$Z(R,L) = \mathrm{Tr} e^{-L\mathscr{H}_R} \quad \text{(L channel)}$$
$$= \mathrm{Tr} e^{-R\mathscr{H}_L} \quad \text{(R channel)}$$

$$\mathscr{H}_R = \frac{1}{2\pi}\int T_{yy}\, dx$$
$$\mathscr{H}_L = \frac{1}{2\pi}\int T_{xx}\, dy$$

(x : coordinate along R axis, y: coordinate along L axis)



TBA for ground state

- $L \to \infty$:

$$\mathrm{Tr}\, e^{-L\mathscr{H}_R} \simeq e^{-LE_0(R)}$$

$$\mathrm{Tr}\, e^{-R\mathscr{H}_L} \simeq e^{-LRf(R)}$$

- R-channel limit: TD limit of an 1D quantum system with $T = R^{-1}$
- TD limit obtained from the Asymptotic Bethe Ansatz (=Bethe-Yang equations) in the limit $L \to \infty$, $N \to \infty$, $N/L \to$ finite.
- from this,

$$f(R) = \mp \frac{1}{R} \sum_{a=1}^{n} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m_a \cosh\theta \log\left[1 \pm e^{-\varepsilon_a(\theta)}\right]$$

- where $\varepsilon_a(\theta)$ is the pseudo-energy, which satisfies

$$\varepsilon_a(\theta) = -m_a R \cosh\theta \mp \sum_{b=1}^{n} \int \frac{d\theta'}{2\pi} \varphi_{ab}(\theta - \theta') \log\left(1 \pm e^{-\varepsilon_b(\theta')}\right)$$

$$\varphi_{ab}(\theta) = -i \frac{d}{d\theta} \log S_{ij}(\theta)$$

Determining the exact spectrum

- 2 methods:
  - attempt to analytically continue the ground state TBA
  - find an integrable lattice regularization and investigate its continuum limit
- For sinh-Gordon model, the exact finite-volume spectrum is known (in the form of integral equations) (proof: Teschner)
- For an N-particle state $\{n_1, n_2, \ldots, n_N\}$, the pseudo-energy is given by

$$\varepsilon(\theta) = mR\cosh\theta + \sum_{j=1}^{N} \log S\left(\theta - \theta_j - \frac{i\pi}{2}\right) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi_{ab}(\theta - \theta') \log\left(1 + e^{-\varepsilon(\theta')}\right)$$

- Quantization condition:

$$\varepsilon\left(\theta_j + \frac{i\pi}{2}\right) = i(2n_j + 1)\pi, \quad j = 1 \ldots N$$

- And the energy is

$$E_N(R) = m\sum_i \cosh\theta_i - m\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \log\left(1 + e^{-\varepsilon(\theta)}\right)$$

Finite volume form factors

- Polynomial corrections (Pozsgay-Takács): take Euclidean two-point function $\langle \mathcal{O}\mathcal{O}' \rangle$ and use that

$$\left\langle \mathcal{O}(\tau,0)\mathcal{O}'(0,0)\right\rangle - \left\langle \mathcal{O}(\tau,0)\mathcal{O}'(0,0)\right\rangle_L \sim O\left(e^{-\mu L}\right)$$

- insert infinite volume resp. finite volume complete system; different integration measures enforce corrections to finite volume matrix elements. The result is

$$\langle 0|\mathcal{O}(0,0)|\{I_1,\dots,I_n\}\rangle_{i_1,\dots,i_n,L} = \frac{F_n^{\mathcal{O}}\left(\tilde{\theta}_1,\dots,\tilde{\theta}_n\right)_{i_1\dots i_n}}{\sqrt{\rho_{i_1\dots i_n}(\tilde{\theta}_1,\dots,\tilde{\theta}_n)}} + O\left(e^{-\mu L}\right)$$

where

$$\rho_{i_1\dots i_n}(\theta_1,\dots,\theta_n) = \det \mathcal{J}^{(n)}(\theta_1,\dots,\theta_n)_{i_1\dots i_n}$$

$$\mathcal{J}^{(n)}_{kl} = \frac{\partial Q_k(\theta_1\dots\theta_n)}{\partial\theta_l}, \quad k,l = 1\dots n$$

and

$$Q_k(\theta_1\dots\theta_n) = m_{i_k} L \sinh\theta_k + \sum_{l\neq k} \delta_{i_k i_l}(\theta_k - \theta_l)$$

Outline

The idea

- The idea is to calculate the **Euclidean torus two-point function** in the limit when one of the radii is sent to infinity:

$$\langle \mathcal{O}(x,t)\,\mathcal{O}\rangle_L = \frac{\int [\mathcal{D}\phi]\,\mathcal{O}(x,t)\,\mathcal{O}(0,0)\,e^{-S[\phi]}}{\int [\mathcal{D}\phi]\,e^{-S[\phi]}}$$

We then use the **finite volume** $\Longleftrightarrow$ **finite temperature** duality

$$\begin{aligned}\langle \mathcal{O}(x,t)\,\mathcal{O}\rangle_L &= \Theta(t)\,\langle 0|\mathcal{O}(x,t)\,\mathcal{O}|0\rangle_L + \Theta(-t)\,\langle 0|\mathcal{O}\,\mathcal{O}(x,t)|0\rangle_L \\ &= \Theta(x)\,\frac{\mathrm{Tr}\left[\mathcal{O}(0,t)\,e^{-H_\infty x}\,\mathcal{O}\,e^{-H_\infty (L-x)}\right]}{\mathrm{Tr}\left[e^{-H_\infty L}\right]} + \\ &\quad \Theta(-x)\,\frac{\mathrm{Tr}\left[\mathcal{O}\,e^{H_\infty x}\,\mathcal{O}(0,t)\,e^{-H_\infty (L+x)}\right]}{\mathrm{Tr}\left[e^{-H_\infty L}\right]}\end{aligned}$$

and extract the relevant information from the poles of the analytically continued momentum-space two-point function

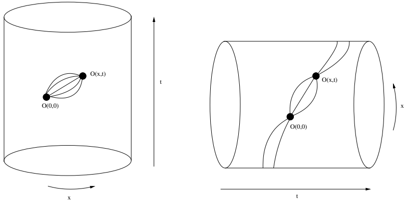
The idea

$$\Gamma(\omega, q) = \frac{1}{L} \int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dt\, e^{i(\omega t + qx)} \langle \mathscr{O}(x,t)\, \mathscr{O} \rangle_L$$

$$= \sum_N |\langle 0 | \mathscr{O} | \theta_1, \ldots, \theta_N \rangle_L|^2 \left\{ \frac{\delta_{q+P_N(L)}}{E_N(L) - i\omega} + \frac{\delta_{q-P_N(L)}}{E_N(L) + i\omega} \right\}$$

$$= \frac{2\pi}{ZL} \sum_{\mu,\nu} |\langle \nu | \mathscr{O} | \mu \rangle|^2 e^{-E_\nu L} \delta(P_\mu - P_\nu + \omega) \left\{ \frac{1}{E_\mu - E_\nu - iq} + \frac{1}{E_\mu - E_\nu + iq} \right\}$$

with $Z = \mathrm{Tr}\left[e^{-H_\infty L}\right]$



The idea

- Observe that:
  - Finite volume energies can be obtained from positions of poles
  - residues are related to finite volume form factors
  - Expansion in ν corresponds to expansion in Lüscher orders
- Exact determination of the two-point function is hopeless, but any systematic expansion leads to a systematic expansion (e.g. in Lüscher orders or coupling) of both the energy levels and form factors.

Realization for $\langle 0|\mathscr{O}|q\rangle_L$

- We will focus on the one-particle finite volume pole

$$\Gamma(\omega,q)=\frac{\mathscr{F}(q)^2}{E(q)+i\omega}+\ldots;\quad \mathscr{F}(q)=\langle 0|\mathscr{O}|q\rangle_L$$

  where $E(q)$ is the exact finite volume energy with momentum $q\equiv m\sinh\theta_1$

- We can expand Γ around the large volume Bethe-Yang pole at $\omega=i\mathscr{E}(q)\equiv im\cosh\theta_1$. At leading Lüscher order, we obtain

$$\Gamma(\omega,q)=\frac{2\pi F_1^2(q)}{L\mathscr{E}(q)}\frac{-i}{\omega-i\mathscr{E}(q)}+\frac{\mathscr{L}_0(q)}{(\omega-i\mathscr{E}(q))^2}+\frac{\mathscr{L}_1(q)}{\omega-i\mathscr{E}(q)}+\text{regular}$$

- Leading exponential corrections read

$$E(q)=\mathscr{E}(q)\left\{1+\frac{L}{2\pi F_1^2}\mathscr{L}_0(q)+\ldots\right\}$$

$$\mathscr{F}(q)=\frac{\sqrt{2\pi F_1}}{\sqrt{L\mathscr{E}(q)}}\left\{1+\frac{iL\mathscr{E}(q)}{4\pi F_1^2}\mathscr{L}_1(q)+\ldots\right\}$$

Realization for $\langle 0 | \mathscr{O} | q \rangle_L$

- Expansion of the partition function:
  $Z = \sum_\nu \langle \nu \mid \nu \rangle e^{-E_\nu L} = 1 + \delta(0) \int_{-\infty}^{\infty} due^{-mL\cosh u} + \ldots$
- Leading (0^{th} order) term in the Lüscher expansion of $\Gamma(\omega, q)$: $|\nu\rangle = |0\rangle$, $|\mu\rangle$ one-particle state
- 1^{st} order term: $|\nu\rangle$ **one-particle state**; $|\mu\rangle = |0\rangle$ or $|\mu\rangle = |\beta_1, \beta_2\rangle$. The potentially singular part can be written

$$\mathscr{L}^{\text{sing}}(\omega, q) = \frac{2\pi}{m^2 L} \int due^{-mL\cosh u} \cdot \left[-\frac{F_1^2 (\delta(0) + \delta(u - \psi))}{\cosh\psi (\cosh\psi - i\hat{q})} + j(u, \psi, q) + (q \leftrightarrow -q) \right]$$

$$j(u, \psi, q) = \int_{-\infty}^{\infty} d\beta_1 \int_{-\infty}^{\beta_1} d\beta_2 \cdot |\langle u | \mathscr{O} | \beta_1, \beta_2 \rangle|^2 \frac{\delta(\sinh\beta_1 + \sinh\beta_2 - \sinh u - \sinh\psi)}{\cosh\beta_1 + \cosh\beta_2 - \cosh u - i\hat{q}}$$

where $q = m\hat{q}$, $\omega = m\sinh\psi$

Realization for $\langle 0 | \mathscr{O} | q \rangle_L$

- Relation of the matrix element $\langle u | \mathscr{O} | \beta_1, \beta_2 \rangle$ to the form factor (from LSZ reduction):

  $$\langle u | \mathscr{O} | \beta_1, \beta_2 \rangle = \delta(u - \beta_1) F_1 + S(\beta_1 - \beta_2) \delta(u - \beta_2) F_1 + F_3(u + i\pi - i\varepsilon, \beta_1, \beta_2)$$

  The integral of its square is divergent and needs to be regularized.

- Regularization:

  $$\delta(x) = \frac{i}{2\pi} \left(\frac{1}{x + i\varepsilon} - \frac{1}{x - i\varepsilon} \right)$$

  (Shown to be equivalent to finite-volume regularization of the form factor)

Regularization and analytic continuation

- We integrate out the δ function in $j(u,\psi,q)$ then shift integration contour. Poles of the regularized matrix elements are then taken into account by the residue theorem. After this, the $\varepsilon \to 0$ limit is performed.
- We then analytically continue $j(u,\psi,q)$ towards $\omega = i\mathscr{E}(q)$.This is done in two steps:
  - First we extend it to a small region where ω is just above the real axis. During this, a double pole will cross the integration contour.
  - After that, the remaining integral is regular; all singular terms are explicit and analytic in ω.

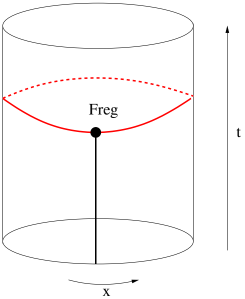
Result

- This method gives back the first Lüscher correction to the one-particle energy (in agreement with TBA).
- For the Lüscher correction of the form factor, we obtain

$$\mathscr{F}(q) = \frac{\sqrt{2\pi}}{\sqrt{\rho_1^{(1)}}} \left\{ F_1 + \int_{-\infty}^{\infty} d\theta F_3^{\mathrm{reg}}\left(\theta + i\pi, \theta, \theta_1^{(0)} - \frac{i\pi}{2}\right) e^{-mL\cosh\theta} + \dots \right\}$$

$$F_3^{\mathrm{reg}}(\theta,\beta_1,\beta_2) = F_3(\theta,\beta_1,\beta_2) - \frac{iF_1\left[1 - S(\beta_1-\beta_2)\right]}{\theta-\beta_1-i\pi} + \frac{iF_1}{2} S'(\beta_1-\beta_2)$$

$$\rho_1 = -i\partial_{\theta_1^{(1)}} \varepsilon^{(1)}\left(\theta_1^{(1)} + \frac{i\pi}{2}\right)$$



Outline

Introduction to integrability

Form factors

Finite volume physics

Nondiagonal 1-particle form factor

Conclusions

Conclusions

- we initiated a programme to calculate systematically both the finite volume energy levels and the finite volume form factors
- performed two different expansions of this finite volume two-point function:
  - finite volume L
  - coupling constant (second order finite-volume Hamiltonian and Lagrangian PT in sinh-Gordon theory)
- These expansions were done explicitely for a moving one-particle state
- Energy expansion was also compared to excited state TBA result
- Future: Lüscher corrections for many-particle states

Conclusions

Thank you for your attention