LeClair-Mussardo theorem and correlation functions in integrable systems

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November 18, 2020

Green-Kubo theorem: in the linear response approximation kinetic coefficients $\{\sigma\}$ can be calculated via the correlataion functions

$$\sigma(q,\omega) = \frac{1}{\omega V}\int_0^\infty dt\, e^{i\omega t}\left\langle \left[J^\dagger(q,t), J(q,0)\right]\right\rangle + i\frac{ne^2}{\omega}.$$

The problem is reduced to the computation of correlation functions

$$\langle J^\dagger(x,t)J(0,0)\rangle_\beta = \sum_n e^{-\beta E_n}\langle\psi_n|J^\dagger(x,t)J(0,0)|\psi_n\rangle$$

Where denotes $|\psi_n\rangle$ secondary quantized n-particles wave functions. If $|\psi\rangle$ are known (usually they are not) correlators, in principle, can be computed using the normal ordering procedure.

However...

• High energy physics. Typical situation is a colliding experiment: pairwise scattering of two free (at large distance) particles, interaction only the short range. Convenient method: switching to the interaction representation, expansion in the series using Feynman diagram technique

• Condensed matter physics. Many particles problem. Convenient method: effectively restriction to a few-particles problem, the rest particles are the background and are described by the additional effective fields or neglected. Example: superconductivity, thus, electrons are organized in the Cooper pairs, interaction with other electrons is neglected, interaction with ions are described by the phonon field

• The exact computation of observables between eigenvectors that contain large number of (quasi)particles is extremely complicated problem even in case of the simplest, for instance δ-like potential. Moreover the exact wave functions are usually unknown. In condensed matter physics typical number of particles is $10^6 - 10^{10}$

Eigenvectors

There exist 1D systems that can be solved using *Bethe ansatz* (1931), H. Bethe. The k-particles secondary quantazied functions (also called *Bethe vectors*) can be presented as

$$|\psi(u_1,\dots,u_n)\rangle = \int d\bar{x}\,\chi(\bar{u}|\bar{x})\psi^\dagger(x_1)\dots\psi^\dagger(x_n)|0\rangle$$

$\bar{u} = \{u_1,\dots,u_n\}$, $\bar{x} = \{x_1,\dots,x_n\}$ with the following restrictions in the periodical case (considered for simplicity)

• Wave function should satisfy following properties

$$\chi(\bar{x}|u_1\dots,u_j\dots u_k\dots) = -\chi(\bar{x}|u_1\dots,u_k\dots u_j\dots), \qquad j\neq k, \qquad j,k=1,\dots,n,$$
$$\chi(x_1\dots,x_j+L\dots|\bar{u}) = \chi(x_1\dots,x_j\dots|\bar{u}), \qquad j=1,\dots,n.$$

• We do not clarify Hamiltonian explicitly, but suggest that there exist factorization properties of scattering

Using the antisymmetric condition and factorization property we can write $\chi_{\bar{\alpha}}(\bar{z}|\bar{u})$ as

$$\chi(\bar{z}|\bar{u}) = \sum_{\tau \in S_a} A(u_{\tau_1}, \dots, u_{\tau_a}) \prod_{k=1}^{a} \tilde{\chi}(u_{\tau_k}|x_{\sigma_k}), \qquad x_{\sigma_1} < \cdots < x_{\sigma_k}$$

where $\tilde{\chi}(u|x) = \tilde{\chi}(u)e^{ip(u)x}$ are one particle functions (plain waves) with a momentum p and coefficients $A(u_1 \dots u_n)$ satisfy

$$\frac{A(\dots, u_j \dots u_k \dots)}{A(\dots, u_k \dots u_j \dots)} = S(u_j, u_k), \qquad j \neq k, \qquad j, k = 1, \dots, n$$

S is an exponent of the two-particle scattering phase: $S(w,u) = \exp(i\Phi(w,u))$. Using the property of periodicity we fix the following condition called *Bethe equation*

$$e^{-ip(u_j)L} \prod_{s \neq j}^{a} S(u_j, u_s) = 1, \qquad j = 1, \dots, a.$$

• Bethe vectors become the eigenvectors if the set of rapidities $\{\bar{u}\}$ (known also as *Bethe or spectral parameters*) satisfies the system of Bethe equation. Such Bethe vectors are called *on-shell*. Otherwise Bethe vectors are called *off-shell or generic*. *Note:* Bethe equation system and Bethe ansatz in general are more complicated the (quasi)particles have internal degrees of freedom, for instance spin, different type of particles, etc.

• Using the commutation relation commutation relations for fields $\{\psi^{\dagger}_{\alpha}(x), \psi_{\beta}(y)\} = \delta_{\alpha,\beta}\delta(x-y)$ correlation functions can be calculated

• The final expression is quite bulky combinatorial expression on the spectral parameters $\{u\}$. The computatiotn pf the themodynamic limit, i.e. particles number $a \to \infty$, system size $L \to \infty$ such that $a/L = \rho$ (where ρ is density) is obviously not an easy task

• There exist multiple approaches to calculate the thermodynamic limit.

Norm of the eigenvector

$$\langle \psi_n | \psi_n \rangle = \prod_{j>k} \frac{(u_j - u_k)^2 + c^2}{(u_j - u_k)^2} \det G(\bar{u})$$

G_{jk} is a Jacobian of Bethe equations

$$G_{jk}(\bar{u}) = \frac{\partial}{\partial u_j} \left[e^{ip(u_k)} \prod_{\ell} S^{-1}(u_k, u_\ell) \right].$$

We define also $G(\bar{u}_{\mathrm{I}}|\bar{u})$, where $G_{jk}(\bar{u}_{\mathrm{I}}|\bar{u})$ denote a sub-block of matrix $G_{jk}(\bar{u})$ of size $m = \#\bar{u}_{\mathrm{I}}.$, which is build from the columns and rows numerated by $u_j, u_k \in \bar{u}_{\mathrm{I}}$

Connected form factors

Consider a finite part of operator $O(x,y) = J^\dagger(x)J(y)$ **between the on-shell** Bethe vectors (eigenvectors)

$$\mathfrak{F}^c_m(\bar u) = \mathrm{Fin}\lim_{\varepsilon_i\to 0}\langle \bar u + \bar\varepsilon | O(x,y) | \bar u\rangle, \qquad u_1 \to u_1 + \varepsilon_1,\ \ u_2 \to u_2 + \varepsilon_2, \dots$$

Important on-shell condition applied **before the limit**. Define *finite part*

$$\mathrm{Fin}\, X : \qquad \text{part of}\, (X) \sim \varepsilon_j/\varepsilon_k = 0, \qquad j \neq k.$$

Finite size LeClair-Mussardo expansion

$$\langle O(x,y)\rangle_a(\bar u) = \frac{\sum \mathfrak{F}^c_{a-m}(\bar u_{\mathrm{II}}) \det G(\bar u_{\mathrm{I}}|\bar u)}{\det G(\bar u)}$$

The sum is taken over all possible partitions $\bar u \to \{\bar u_{\mathrm{I}}, \bar u_{\mathrm{II}}\}$ including cases $\bar u_{\mathrm{I}} = \varnothing$, $\bar u_{\mathrm{II}} = \varnothing$. Cardinalities are denoted as $\#\bar u_{\mathrm{I}} = m$, $\#\bar u_{\mathrm{II}} = a - m$.

Thermodynamic limit of expansion

A. Leclair, G. Mussardo, *Nucl. Phys. B* **552** (1999), 624–642.
B. Pozsgay, G. Takacs, *Nucl. Phys. B* **788** (2008), 209–251.
The ratio of determinants

$$\frac{\det G(\bar{u}_{\mathrm{I}}|\bar{u})}{\det G(\bar{u})} = \prod_{u_j \in \bar{u}_{\mathrm{II}}} \rho(u_j),$$

ρ is the density function. The thermodynamic limit of the LM expansion

$$\langle O(x,y)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \int \frac{dt_j}{(2\pi)} \mathfrak{F}_n^c(\bar{t}),$$

the integral is taking over the Fermi zone.

- How to compute $\mathfrak{F}_n^c$?
- Whether the series is converge fast enough?

Symmetric form factor expansion

Define a symmetric limit of operator $O(x, y)$ **between the on-shell** Bethe vectors (eigenvectors). Important: **on-shell condition is applied before the limit**.

$$\mathfrak{F}_m(\bar{u}) = \lim_{\varepsilon \to 0} \langle \bar{u} + \varepsilon | O(x, y) | \bar{u} \rangle, \qquad O(x, y) = J^\dagger(x) J(y),$$

i. e. on-shell condition is applied fist for left and right vectors and then the limit of the left state to the right is taken. The finite size expansion is given by

$$\langle O(x, y) \rangle_a (\bar{u}) = \frac{\sum \mathfrak{F}_{a-m} (\bar{u}_{\mathrm{II}}) \det G (\bar{u}_{\mathrm{I}})}{\det G (\bar{u})}$$

The sum is taken over all possible partitions $\bar{u} \to \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$ including cases $\bar{u}_{\mathrm{I}} = \varnothing$, $\bar{u}_{\mathrm{II}} = \varnothing$. Cardinalities are denoted as $\#\bar{u}_{\mathrm{I}} = m$, $\#\bar{u}_{\mathrm{II}} = a - m$.

B. Pozsgay, *J. Stat. Mech.* (2011), P01011,

B. Pozsgay, G. Takacs, *J. Stat. Mech.* **11** (2010), 12.

Thermodynamic limit

The limit of two Gaudin determinants is given by

$$\frac{\det G(\bar{u}_{\mathrm{I}})}{\det G(\bar{u})} = \prod_{u_j \in \bar{u}_{\mathrm{II}}} \omega(u_j)\rho(u_j), \qquad \omega(u) = \exp\left(-\frac{1}{2\pi}\int_{-Q}^{Q} dv\, K(v,u)\right),$$

$K(x,y) = \partial_x S(x,y)$, $\bar{u} = \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$, $\pm Q$ are Fermi boundaries, ρ is a density function of Bethe parameters. Hereby the thermodynamic limit of LM expansion can be derived

$$\langle O(x,y)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \int \frac{dt_j}{(2\pi)} \omega(t_j) \mathfrak{F}_n(\bar{t}),$$

The similar expansion for Bose gas was made by
D. B. Creamer, H. B. Thacker, D. Wilkinson, *Phys. Rev. D* **23** (1981)

- How to compute $\mathfrak{F}_n$?
- Whether the series is converge fast enough?

Irreducible parts

Consider the limit with fixed momenta $p(u)$. Important: **on-shell condition is applied before the limit**.

$$I_a\left(\bar{u}\right) = \lim_{\varepsilon\to 0} \langle \bar{u}+\varepsilon|O(x,y)|\bar{u}\rangle\big|_{e^{ip(u_j)x}=const.},$$

Expand I_a in the Fourier series $\mathcal{A}_n$

$$I_a(\bar{u}) = \sum \left(\prod_{j=1}^{m} e^{i\left(p(u_j^-)-ip(u_j^+)\right)x} - 1 \right) \mathcal{A}_n\left(\bar{u}^+;\bar{u}^-;\bar{u}^0\right).$$

where p is momenta, the sum is taken over all possible partitions $\bar{u}\to\{\bar{u}^+,\bar{u}^-,\bar{u}^0\}$, $n=\#\bar{u}+=\#\bar{u}^-$ including cases $\bar{u}^+=\bar{u}^-=\varnothing$ or $\bar{u}^0=\varnothing$

Irreducible parts expansion

The following expansion can be established

$$\langle J_\alpha(x) J_\beta(0)\rangle = \langle J_\alpha\rangle\langle J_\beta\rangle + \sum_{k=2}^{\infty}\Gamma_k.$$

$$\Gamma_k = \frac{1}{k!}\prod_{j=1}^{k}\int \frac{dt_j}{(2\pi)}\omega(t_j)\sum \mathcal{A}_k\left(\bar{t}^+;\bar{t}^-;\bar{t}^0\right)\exp\left(ixp_n(\bar{t}^+;\bar{t}^-)\right)$$

the sum is taken over all possible partitions of $\bar{t}\to\{\bar{t}^+,\bar{t}^-,\bar{t}^0\}$ including cases where $\bar{t}^+ = \bar{t}^- = \varnothing$ or $\bar{t}^0 = \varnothing$. $P_n(\bar{t}^+,\bar{t}^-)$ is a renormalised momentum

$$1 + P_n(s|\bar{u}^+;\bar{u}^-) = \prod_{j\in\bar{u}^+}\frac{f(u_j,s)}{f(s,u_j)}\prod_{k\in\bar{u}^-}\frac{f(s,u_k)}{f(u_k,s)}\exp\left(\int_{-Q}^{Q} K(p,s)P_n(s|\bar{u}^+;\bar{u}^-)ds\right)$$

Note: here only Fourier coefficients depend on particular operators J_α.

Physical system examples

Lieb-Liniger model (1D bose gas), E. H. Lieb, W. Liniger, *Phys. Rev.* **130** (1963)

$$\mathcal{H} = \int_0^L dx \left\{ -\psi^\dagger(x)\partial_x^2\psi(x) + \varkappa\left(\psi(x)^\dagger\psi(x)\right)^2 \right\},$$

Spin-1/2 1D Fermi gas, C. N. Yang, *Phys. Rev. Lett.* **19** (1967)

$$\mathcal{H} = \int_0^L dx \left\{ \partial\psi_\alpha^\dagger\partial\psi_\alpha + \varkappa\psi_\alpha^\dagger\psi_\beta^\dagger\psi_\beta\psi_\alpha \right\}, \qquad \alpha,\beta = \uparrow,\downarrow,$$

$$\left\{ \psi_\alpha^\dagger(x), \psi_\beta(y) \right\} = \delta_{\alpha\beta}\delta(x-y).$$

Y.-K. Zhou, B.-H. Zhao, *Phys. Lett. A.* **123** (1987).

V. E. Korepin, *Comm. Math. Phys.* **94** (1984), 93.

$$\langle n(x)n(0)\rangle = \frac{Q^2}{\pi^2} + \frac{4Q^2}{c\pi^3} + \left(1 + \frac{2}{c}\frac{\partial}{\partial x_r}\right)\frac{\sin^2(Qx_r)}{2\pi^2 x_r^2} - \frac{4Q}{\pi c}\frac{\sin^2(Qx_r)}{2\pi^2 x_r^2}$$
$$+\frac{1}{c}\frac{4Q}{(2\pi)^3}\frac{\sin^2(Qx)}{x^2} + \frac{2}{c}\frac{\partial}{\partial x}\sin(Qx)\int_{-Q}^{Q}\frac{du}{(2\pi)^3}\sin(ux)\ln\left(\frac{Q+u}{Q-u}\right)$$

Q is a Fermi boundary for the Bose gas, $x_r = x(1 + 2Q/(\pi c))$.
In the same way it is easy to derive correlators in Fermi gas, where $n_{\uparrow\downarrow}$ are densities with the correspondent spin projections, $n = n_\uparrow + n_\downarrow$

$$\langle n(x)n(0)\rangle = \left(\frac{Q^2}{\pi^2} - \frac{\sin^2(Qx)}{x^2\pi^2}\right) + O(1/c^2).$$

$$\langle n_\uparrow(x)n_\downarrow(0)\rangle = \frac{4Q^2B}{\pi^3 c} - \frac{4B\sin^2(Qx)}{\pi^3 x^2 c} + O(1/c^2).$$

Q is a Fermi boundary for the Fermi gas and $B = n_\uparrow \pi^2 c/(4Q)$

Algebraic Bethe ansatz (ABA)

The approach to integrability can be reformulated via the following axioms
• *R-matrix* satisfies *Yang-Baxter equation*

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

Equation holds in a tensor product of spaces $\mathbb{C}^{\otimes 3}$. Subscripts denote the number of space in which R_{jk} acts nontrivially.
• The system is provided by the *Lax operator* $L(v)$ that depends on *spectral parameters* λ, μ and satisfies *RLL*-relation

$$R_{12}(u-v)L_{13}(u)L_{23}(v) = L_{23}(v)L_{13}(u)R_{12}(u-v).$$

Equation holds in a tensor product of spaces $\mathbb{C} \otimes \mathbb{C} \otimes \mathcal{H}$, $\mathcal{H}$ is a Hilbert space of the one-site Hamiltonian, $\mathbb{C}$ is called *an auxiliary space*. Subscripts denote the numbers of spaces in which R_{jk} acts nontrivially.

• *Monodromy matrix* (the name is taken from the classical integrability) is expressed via the Lax matrix as

$$T(u) = L_N(u) \cdots L_1(u).$$

• All the information about the system is included in the monodromy matrix. Thus, physical operators can be expressed via the monodromy matrix entries

• Trace of T matrix is called *transfer matrix*. $t(u)$ is the generation function the **pairwise commuting** integrals of motion $\{Q_k\}$ (natural extension of the classical definition of integrability). Take the trace over the auxiliary space

$$tr_0 \, T_{01}(u) = \sum_k Q_k (u - u_0)^k.$$

Hamiltonian included in this scheme, for example in considered cases as Q_2

Physical system examples

- Discrete version of 1D Bose gas with nonlinear interaction and lattice spacing Δ and lattice defined fields ψ_n, $\psi_n^\dagger$

$$L_n(u) = \frac{1}{1 - \frac{\mathrm{i}u\Delta}{2}} \begin{pmatrix} 1 - \frac{\mathrm{i}u\Delta}{2} & -\mathrm{i}\sqrt{c}\Delta\psi_n^\dagger \\ \mathrm{i}\sqrt{c}\Delta\psi_n & 1 + \frac{\mathrm{i}u\Delta}{2} \end{pmatrix} + O(\Delta^2),$$

with the Hamiltonian in the continual limit (spacing $\Delta \to 0$, number of sites $N \to \infty$ while $N\Delta \to L$, where L is a system length)

$$\mathcal{H} = \int_0^L dx \left\{ -\psi^\dagger(x)\partial_x^2\psi(x) + 2c\left(\psi(x)^\dagger\psi(x)\right)^2 \right\},$$

Algebraic Bethe ansatz

- Vacuums, creation and annihilation operators

$$T(u) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix},$$

$$\langle 0| T_{ij}(u) = 0, \qquad T_{ij}(u)|0\rangle = 0, \qquad \langle 0|0\rangle = 1.$$

- Left and right vacuums $\langle 0|$ and $|0\rangle$ are eigenvectors of the diagonal matrix elements

$$T_{ii}(u)|0\rangle = \Lambda_i(u)|0\rangle, \qquad \langle 0| T_{ii}(\lambda) = \langle 0|\Lambda_i(u).$$

- *Bethe vectors* are given by special polynomials on the monodromy matrix entries that act onto vacuum $|0\rangle$

$$|\bar{u}\rangle = |u_1, \cdots, u_a\rangle = Pol(T_{ij}(u_k))|0\rangle,$$
$$\langle \bar{u}| = \langle u_1, \cdots, u_a)| = \langle 0|Pol(T_{ij}(u_k))|.$$

- Expressing explicitly matrix elements T_{jk} in the Bethe vector it is easy to check by direct comparison that Bethe vectors defined via the monodromy matrix entries directly coincide with one given in initial formulation Bethe ansatz
- ABA is technically simplification in multiple application. Thus it is much simple to compute irreducible parts or symmetric form factors using ABA
- A lot of computation are depending only on the symmetry of R (for instance Lieb-Liniger has algebra symmetry $\mathfrak{gl}(2)$, Gaudin-Yang model has symmetry $\mathfrak{gl}(2|1)$)
- General approach to the integrability: we know that the system is integrable and how to solve it if there exists Lax matrix
- Also it is clear how explicitly build the integrals of motion

Conclusions

- Dynamical case
- Finite temperature correlators
- Models with more internal degrees of freedom: i.e. models whose R-matrix has algebra symmetry different from the simplest cases $\mathfrak{gl}(2)$, $\mathfrak{gl}(2|1)$, $\mathfrak{gl}(3)$
- Expansion that will be valid for arbitrary particles densities and coupling constant