Strong coupling results from the numerical study of the spectrum in planar N=4 SYM theory

Based on:

N.Gromov,Á. Hegedűs, J. Julius,N. Sokolova:

arXiv:2306.12379

Motivation:

N=4 SYM theory is a conformal field theory, thus anomalous dimensions and OPE coeffcients are of primary importance for solving the theory

In the planar limit, with the help of integrability , we determined numerically hundreds of anomalous dimensions with high precision in a wide range of 't Hooft coupling, because:

- In the lack of analytical methods, the strong coupling behaviour of anomalous dimensions can be explored only numerically

- Anomalous dimensions are important input data for numerical Conformal Bootstrap methods to constrain the OPE coefficients of the theory

Outline

- Introduction : planar AdS/CFT spectral problem
- From integrability to Quantum Spectral Curve eqs. (QSC)
- QSC: a short summary
- Numerical solution of QSC
- Strong coupling results from numerical data

Anomalous dimensions

constraining OPE coefficients

- Conclusion and summary

AdS/CFT correspondence

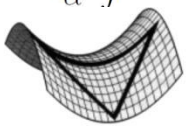
Type-II B string theory on $AdS_5 \times S^5$ background

=

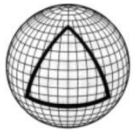
N=4 Super Yang-Mills theory with $SU(N_c)$ gauge group

$$S = \frac{R^2}{\alpha} \int d\sigma d\tau \, G_{\mu\nu}(X) \, \partial^a X(\sigma,\tau) \, \partial_a X(\sigma,\tau) + \text{ fermions}$$

$$S = \frac{2}{g^2} \int d^4x \, \mathrm{Tr}\{-\tfrac{1}{4}F^2 - \tfrac{1}{2}(D\phi)^2 + i\,\bar{\psi} D\psi + \tfrac{1}{2}[\phi,\phi]^2 + \bar{\psi}[\phi,\psi]\}$$



isometry of AdS_5 : SO(4,2) $\longrightarrow$ conformal symmetry



isometry of S^5 : SO(6) $\longrightarrow$ rotation of 6 scalars

} PSU(2,2|4)

Parameters

$$\lambda = g_{YM}^2 N_c$$

't Hooft coupling

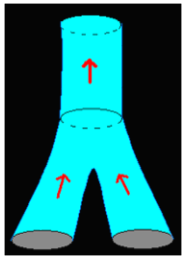
$$\sqrt{\lambda} = \frac{R^2}{\alpha}$$

Radius of $AdS_5 \times S^5$

String tension

$$g_s = \frac{\lambda}{N_c}$$

String coupling



Dictionary

String energies $\longleftrightarrow$ anomalous dimensions

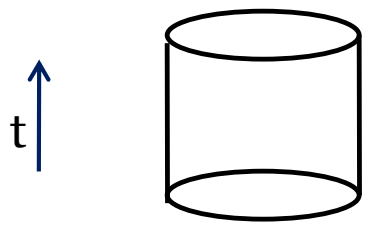
- Spectral problem of AdS/CFT

AdS/CFT: planar limit

- Planar limit: $N_c \to \infty$ ⟹ $g_s \to 0$ No string-interactions

- Integrability on both sides of the correspondence

String theory:
1+1 dim. integrable
σ-model on a cylinder



Fields:

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$$

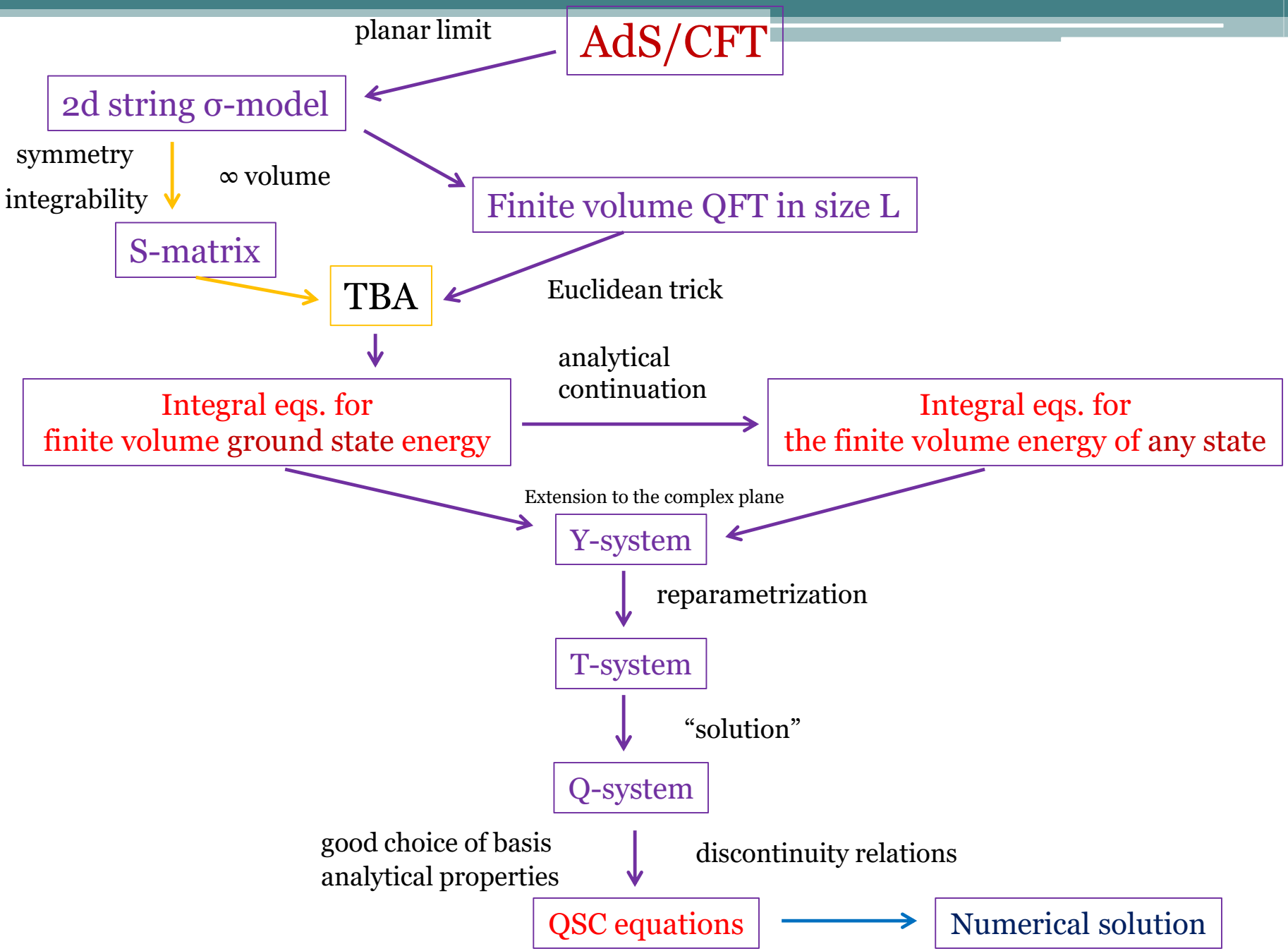
SYM:
Single trace operators:

$$\mathrm{tr}\,\mathcal{W}_1\,\ldots\,\mathcal{W}_L$$

$$\mathcal{W}_I \in \left\{\mathcal{D}^i\,\Phi\,,\mathcal{D}^j\,\Psi\,,\mathcal{D}^k\,\bar{\Psi}\,,\mathcal{D}^l\,\mathcal{F}\right\}$$

PSU(2,2|4) long range spin-chain

Dilatation operator ↔ Hamiltonian


AdS/CFT
planar limit
2d string σ-model
symmetry
integrability
∞ volume
Finite volume QFT in size L
S-matrix
TBA
Euclidean trick
Integral eqs. for
finite volume ground state energy
analytical
continuation
Integral eqs. for
the finite volume energy of any state
Extension to the complex plane
Y-system
reparametrization
T-system
"solution"
Q-system
good choice of basis
analytical properties
discontinuity relations
QSC equations
Numerical solution

Minimizing the free energy

$$\frac{\delta}{\delta\rho}(E - T\cdot S) = 0$$

$$\frac{\delta}{\delta\rho}\Big(\sum_n \epsilon(p_n) - T\cdot(\text{number of microstates}) + \lambda\cdot(\text{ABA eqs.})\Big) = 0$$

- Complicated set of TBA integral equations

$$\ln Y_A(u) = \epsilon_A(u) + K_{AB}\star\ln(1+Y_B) \qquad \ln Y_A(u) \sim \frac{\rho_a^{(h)}(u)}{\rho_a(u)}$$

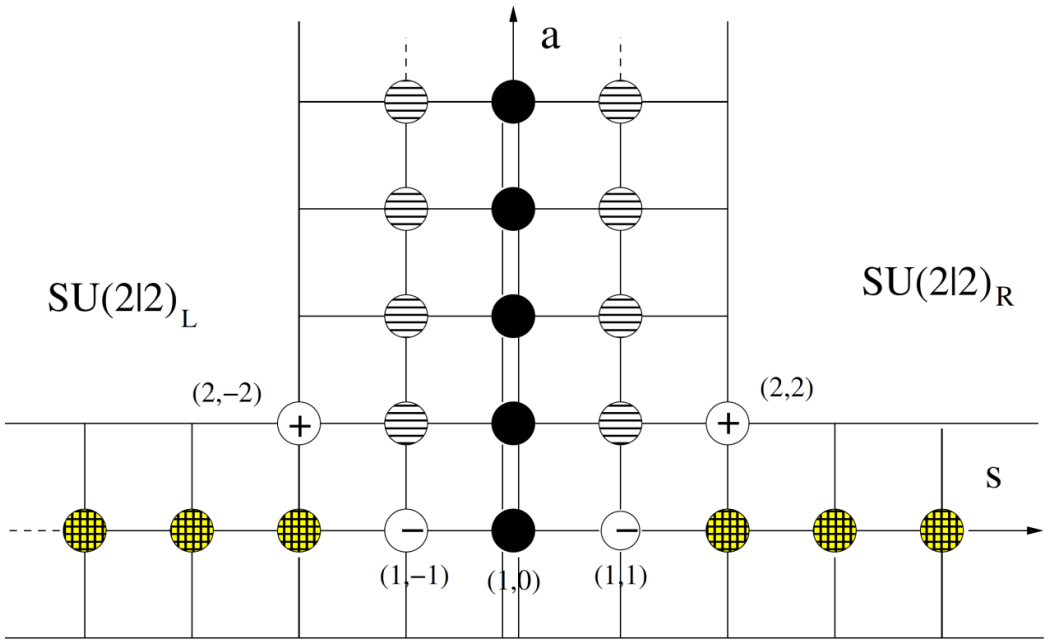
$$E_0(L) = \sum_A \int du\epsilon_A(u)\,\ln(1+Y_A(u))$$

Important property: Y-system equations

$$Y_{a,s}(u+i/2)\,Y_{a,s}(u-i/2) = \frac{(1+Y_{a,s-1}(u))\,(1+Y_{a,s+1}(u))}{\left(1+\frac{1}{Y_{a-1,s}(u)}\right)\left(1+\frac{1}{Y_{a+1,s}(u)}\right)}$$

Excited states

- From analytical continuations of the ground state eqs.
- Solutions of the Y-system with specific analytical properties
- AdS/CFT:



- Domain: reflects the PSU(2,2|4) symmetry

Reduction of the # of variables

- AdS/CFT Y-system is hard to treat

- Find new variables in which the problem is simpler

- Y → T: $$Y_{a,s} = \frac{T_{a,s-1}\,T_{a,s+1}}{T_{a-1,s}\,T_{a+1,s}}$$

$$T_{a,s}(u+i/2)\,T_{a,s}(u-i/2) = T_{a,s-1}(u)\,T_{a,s+1}(u) + T_{a-1,s}(u)\,T_{a+1,s}(u)$$

- T encodes the global symmetry of the model

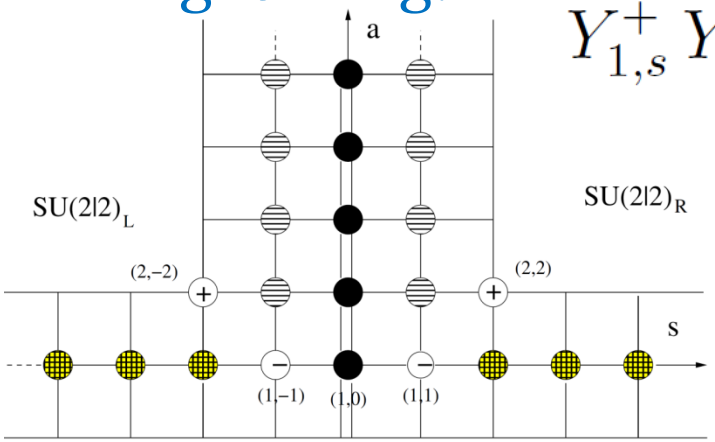
Characters of representations belonging to (a,s) rectangular Young-tablouxs satisfy constant T-system eqs.

- T → Q: finite # of variables

How to imagine T→Q?

$$f(u \pm i\,n/2) = f^{[\pm n]}(u)$$

- Right-wing:



$$Y^{+}_{1,s}\,Y^{-}_{1,s} = (1 + Y_{1,s+1})\,(1 + Y_{1,s-1}) \qquad s \geq 2$$

$$T^{+}_{1,s}\,T^{-}_{1,s} = T_{1,s-1}\,T_{1,s+1} + T^{[s]}_{1,1}\,T^{[-s]}_{1,1}$$

T→Q is similar to solving "a discrete version" of Laplace-equation:

$$T^{+}_{s}\,T^{-}_{s} = T_{s-1}\,T_{s+1} \quad \Rightarrow \quad T_s(u) = f_L(u - i\,\tfrac{s}{2})\,f_R(u + i\,\tfrac{s}{2})$$

Here:

$$T_{1,s}(u) = P_1(u + i\,\tfrac{s}{2})\,P_2(u - i\,\tfrac{s}{2}) - P_1(u - i\,\tfrac{s}{2})\,P_2(u + i\,\tfrac{s}{2})$$

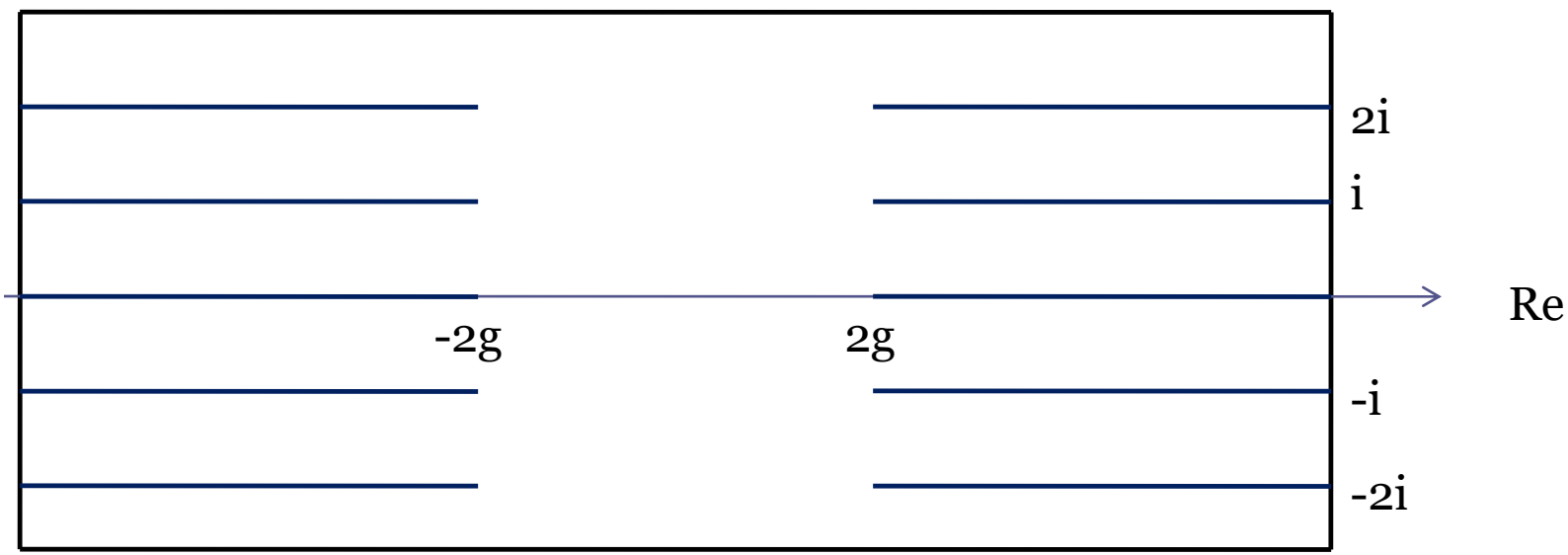
Discontinuities

- AdS/CFT: discontinuity relations $\mathrm{disc}(\ln Y)_n = \mathcal{F}_n(Y)$

$$\mathrm{disc}\, f_n(u) = f(u + i\,n + i\,0) - f(u + i\,n - i\,0)$$

Branch points: $u = \pm 2g + i\mathbb{Z}$ $\qquad g = \frac{\sqrt{\lambda}}{4\pi}$

Y-syst.+disc. rel.+analyticity $\Longrightarrow$ TBA eqs.



Q-system

Gromov, Kazakov, Leurent, Tsuboi '11

- T-hook: 2^8 Q-functions $Q_{A|I}(u)$ $\quad A=\{a_1,a_2,.....\}$ $\quad I=\{i_1,i_2,...\}$ $\quad a,i=1,2,3,4$

$$Q_{...ab...|...ij...} = -Q_{...ba...|...ij...} = Q_{...ba...|...ji...}$$

- 8 independent:

$$Q_{A|I}Q_{Aab|I} = Q^+_{Aa|I}Q^-_{Ab|I} - Q^-_{Aa|I}Q^+_{Ab|I}$$
$$Q_{A|I}Q_{A|Iij} = Q^+_{A|Ii}Q^-_{A|Ij} - Q^-_{A|Ii}Q^+_{A|Ij}$$
$$Q_{Aa|I}Q_{A|Ii} = Q^+_{Aa|Ii}Q^-_{A|I} - Q^+_{A|I}Q^-_{Aa|Ii}$$

- Symmetries:

Hodge-transformation:

$$Q^{A|I} \equiv (-1)^{|A'||I|}\epsilon^{A'A}\epsilon^{I'I}Q_{A'|I'} \qquad \{A'\}=\{1,2,3,4\}\setminus\{A\} \qquad \{I'\}=\{1,2,3,4\}\setminus\{I\}$$

H-symmetry:

$$Q_{A|I} \to \sum_{\substack{|B|=|A|\\|J|=|I|}} (H_b^{[|A|-|I|]})_A{}^B (H_f^{[|A|-|I|]})_I{}^J Q_{B|J} \qquad H^+ = H^-$$

Analyticity:

$$Q^{\mathrm{LHPA}} = H\cdot Q^{\mathrm{UHPA}} \qquad H^+ = H^-$$

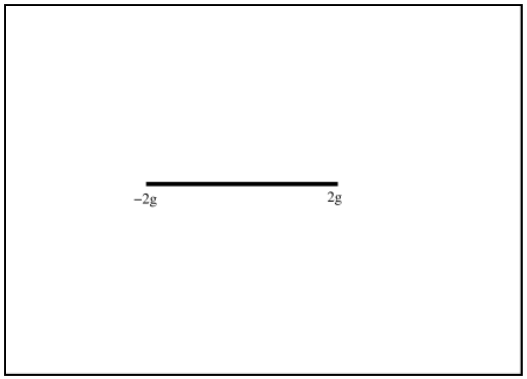
QSC equations 1.

Gromov, Kazakov, Leurent, Volin '13

Basis of Q-functions:

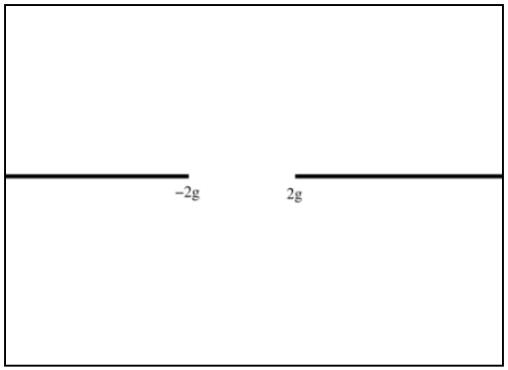
$Q_{a|\emptyset} \rightarrow$ P_a, P^a, a=1,..4

$Q_{\emptyset|i} \rightarrow$ Q_i, Q^i, i=1,..4



Systems

$$g = \frac{\sqrt{\lambda}}{4\pi}$$



Pμ

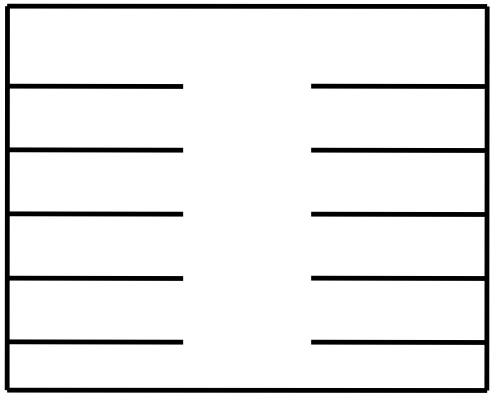
Qω

μ_{ab}

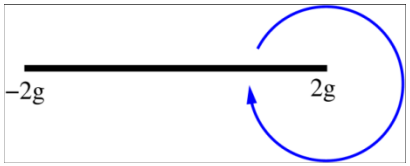
Branch points of square root type:

$$u = \pm 2g + i\mathbb{Z}$$

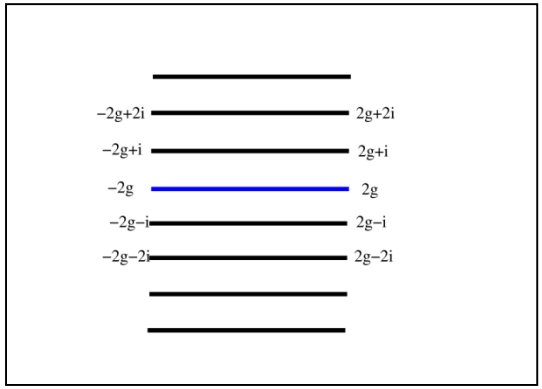
ω_{ij}



$$f(u) \rightarrow \tilde{f}(u)$$



$$\tilde{\tilde{f}}(u) = f(u)$$



QSC equations 2.

$$\mathbf{P}^a\,\mathbf{P}_a = 0, \qquad a = 1, \ldots, 4$$

$$\mu_{ab} - \tilde{\mu}_{ab} = \tilde{\mathbf{P}}_a\,\mathbf{P}_b - \tilde{\mathbf{P}}_b\,\mathbf{P}_a,$$

$$\tilde{\mathbf{P}}_a = \mu_{ab}\,\mathbf{P}^b,$$

$$\mu_{ab}(u) = \mu_{ab}(u+i),$$

$$\mathrm{Pf}(\mu) \equiv \mu_{12}\mu_{34} - \mu_{13}\mu_{24} + \mu_{14}\mu_{23} = 1.$$

$$\mu_{ab} = -\mu_{ba}$$

$$\mathbf{Q}^i\,\mathbf{Q}_i = 0, \qquad i = 1, \ldots, 4,$$

$$\omega_{ij} - \tilde{\omega}_{ij} = \tilde{\mathbf{Q}}_i\,\mathbf{Q}_j - \tilde{\mathbf{Q}}_j\,\mathbf{Q}_i,$$

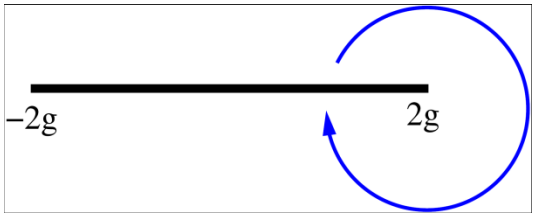
$$\tilde{\mathbf{Q}}_i = \omega_{ij}\,\mathbf{Q}^j,$$

$$\omega_{ij}(u) = \omega_{ij}(u+i),$$

$$\mathrm{Pf}(\omega) = 1.$$

$$\omega_{ij} = -\omega_{ji}$$

$$f(u) \to \tilde{f}(u)$$



$\mathbf{P}_a,\ \mu_{ab}$ have no poles!

$\mathbf{Q}_i,\ \omega_{ij}$ have no poles!

QSC equations 3.

- Unique solution: analyticity + large u asymptotics

$$\mathbf{P}_a \sim A_a u^{-\tilde{M}_a}, \qquad \mathbf{Q}_i \sim B_i u^{\hat{M}_i - 1},$$

$$\mathbf{P}^a \sim A^a u^{\tilde{M}_a - 1}, \qquad \mathbf{Q}^i \sim B^i u^{-\hat{M}_i}$$

$$\mu_{12} \sim u^{\Delta - L}, \ \mu_{13} \sim u^{\Delta - 1}, \quad \text{etc.} \qquad \omega_{ij} \to \text{const.}$$

$$\hat{M}_i = \left\{1 + \frac{\Delta - \ell_1}{2}, 2 + \frac{\Delta + \ell_1}{2}, -1 - \frac{\Delta + \ell_2}{2}, -\frac{\Delta - \ell_2}{2}\right\}$$

$$\tilde{M}_a = \left\{2 - \frac{p}{2} - \frac{3\,q_1 + q_2}{4}, 1 - \frac{p}{2} + \frac{q_1 - q_2}{4}, \frac{p}{2} + \frac{q_1 - q_2}{4}, -1 + \frac{p}{2} + \frac{q_1 + 3\,q_2}{4}\right\}$$

$$[\underbrace{\ell_1 \ \ell_2}_{\text{Lorentz spins}} \ \underbrace{q_1 \ p \ q_2}_{\text{SO(6) labels}}]$$

Characterization of states

- States form multiplets with respect to the full psu(2,2|4) symmetry of the theory

- Multiplets are characterized by their g=0, quantum numbers:

$$[\underbrace{\ell_1 \;\; \ell_2}_{\text{Lorentz spins}} \;\; \underbrace{q_1 \;\; p \;\; q_2}_{\text{SO(6) labels}}] \qquad \Delta_0 \qquad B \qquad L$$

Δ_0 → classical dimension

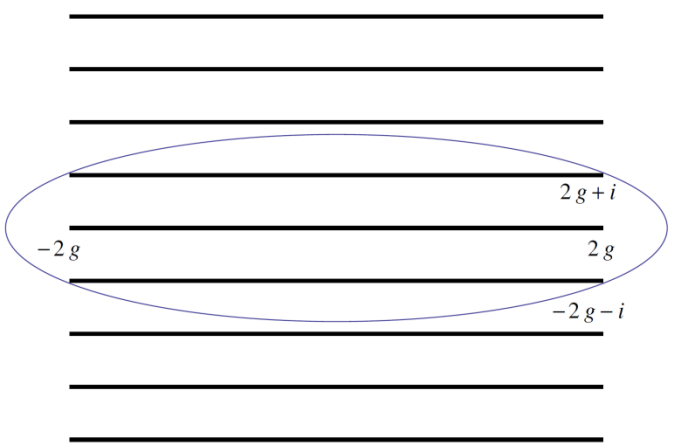
B → Hyper-charge

L → Length

- All degeneracies are lifted only by quantum corrections

Numerical solution of QSC '15 Gromov et al.

- Parametrization:



Zhukovsky-variable: $x + \frac{1}{x} = \frac{u}{g}$: $\tilde{x} = \frac{1}{x}$

$$\mathbf{P}_a(u) = \sum_{n=0}^{\infty} \frac{c_{a,n}(g)}{x(u)^{\tilde{M}_a+n}}$$

short cut version: $|x| > 1$

$$\tilde{\mathbf{P}}_a(u) = \sum_{n=0}^{\infty} c_{a,n}(g)\, x(u)^{\tilde{M}_a+n}$$

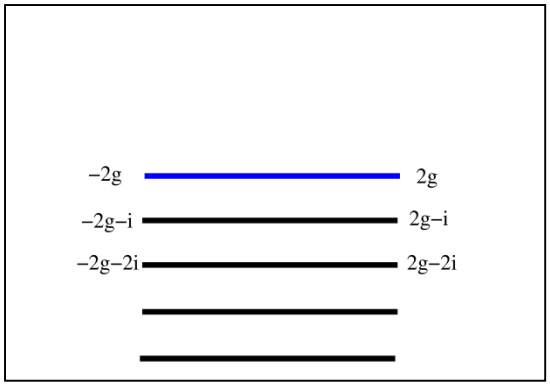
- Radius of convergence : $R(g) = |x(2g+i)|$
- At weak coupling: $c_{a,n}(g) \sim g^{|n|}$
- If the loop order is fixed only a finite # of coeffs. contribute

Numerical solution of QSC 2.

- 1st step: $P_a \to Q_i,\ \tilde{Q}_i$ through $Q_{a|i}$ UHPA

Solve: $\mathcal{Q}_{a|i}(u+\frac{i}{2}) - \mathcal{Q}_{a|i}(u-\frac{i}{2}) = -\mathbf{P}_a(u)\mathbf{P}^b(u)\,\mathcal{Q}_{b|i}(u+\frac{i}{2}) \qquad a,i \in \{1,2,3,4\}.$

Then:
$$\left.\begin{aligned} \mathbf{Q}_i(u) &= -\mathbf{P}^a(u)\ \mathcal{Q}_{a|i}(u+i/2) \\ \tilde{\mathbf{Q}}_i(u) &= -\tilde{\mathbf{P}}^a(u)\ \mathcal{Q}_{a|i}(u+i/2). \end{aligned}\right\} \mathrm{Im}u > 0$$



- 2nd step: close the equations by gluing conditions:

$$\tilde{\mathbf{Q}}_i = G_{i\,j}\,\bar{\mathbf{Q}}^j\ ,$$
$$\tilde{\mathbf{Q}}^i = G^{i\,j}\,\bar{\mathbf{Q}}_j\ ,$$

$$G_{i\,j} = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1 \\ 0 & 0 & \beta_2 & 0 \end{pmatrix}_{i\,j}$$

$$\alpha_1 = \bar{\alpha}_2$$
$$\beta_1 = \bar{\beta}_2$$

Numerical solution of QSC 3.

- In reality the gluing equations are imposed in a " backward" transformed form: $Q_i \longrightarrow P_a \longrightarrow c_{a,n}$

Define a new P: $$\mathbf{P}'_a(u) = -Q^+_{a|k}\, G^{kj}\, \bar{\mathbf{Q}}_j(\{c\}, u)$$

Gluing conditions imply: $$\mathbf{P}'_a(u) = \mathbf{P}_a(u)$$

In numerics the equality of the coefficients are imposed:

$$c_{a,n} = c'_{a,n}(\{c\}) \qquad \mathbf{P}'_a(u) = \sum_{n=0}^{\infty} \frac{c'_{a,n}(g)}{x(u)^{\tilde{M}_a+n}}$$

Success of QSC

QSC: nonlinear Riemann-Hilbert problem for a few unknowns

Very efficient!

- 11-loop results at weak coupling '18 Marboe,Volin
- Bremstrahlung-function for the quark-anti-quark potential upto $(\theta-\varphi)^2$ order '15 Gromov,Levkovich-Mashlyuk
- Small spin expansion upto $O(S^2)$ '14 Gromov,L-Mashlyuk, Sizov, Valatka
- NNLO results in the BFKL ($S \to -1$) limit '14 Alfimov,Gromov, Kazakov
  '15 Gromov,L-Mashlyuk, Sizov
- Strong coupling solution still lacks!

Numerical work

- We made a publicly available C++ implementation of the numerical algorithm

- We determined Δ for all 219 states with $\Delta_0 \leq 6$

in a wide range of the 't Hooft coupling:

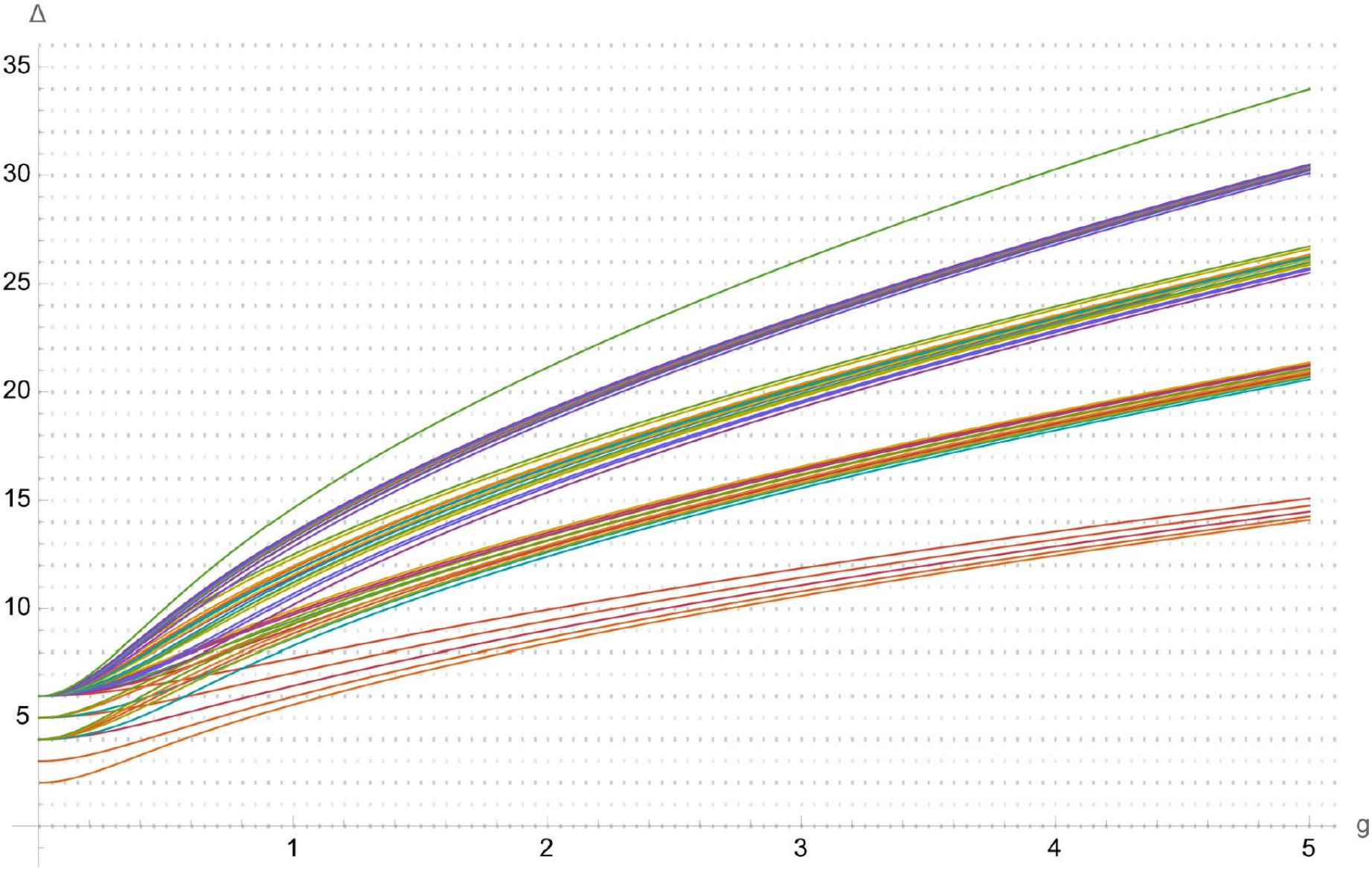
$0 \leq g \leq g_{max}$ with $g_{max} = 2, 5$ depending on the state

I.e: $0 \leq \lambda \leq \lambda_{max}$ with $\lambda_{max} \approx 630, 4000$

We determined some leading strong coupling expansion coefficients of these Δs

We analyzed the results

Numerical results



Testing some know formulas

There are predictions for the minimal anomalous dimension for states with quantum numbers:

[S-2, S-2, 0, L-2, 0] Δ_0=L+S-2

- Δ at strong coupling:

$$\Delta = \Delta^{(0)}\lambda^{\frac{1}{4}} + \Delta^{(1)}\lambda^{-\frac{1}{4}} + \Delta^{(2)}\lambda^{-\frac{3}{4}} + \Delta^{(3)}\lambda^{-\frac{5}{4}} + +\Delta^{(4)}\lambda^{-\frac{7}{4}} + \Delta^{(5)}\lambda^{-\frac{9}{4}} + \ldots$$

'11 Basso
'11 Gromov , Serban, Shenderovich, Volin
'11 Roiban, Tseytlin , '11 Vallilo, Mazzucato
'14 Gromov, L-Maslyuk, Sizov, Valatka

- Analytical predictions:

$$\Delta^{(0)} = \sqrt{2\,S}, \qquad \Delta^{(1)} = \frac{2\,L^2 + S(3\,S-2)}{4\sqrt{2S}},$$

$$\Delta^{(2)} = \frac{-21\,S^4 + (24 - 96\,\zeta_3)S^3 + 4\,(5L^2-3)\,S^2 + 8L^2S - 4L^4}{64\sqrt{2}\,S^{3/2}},$$

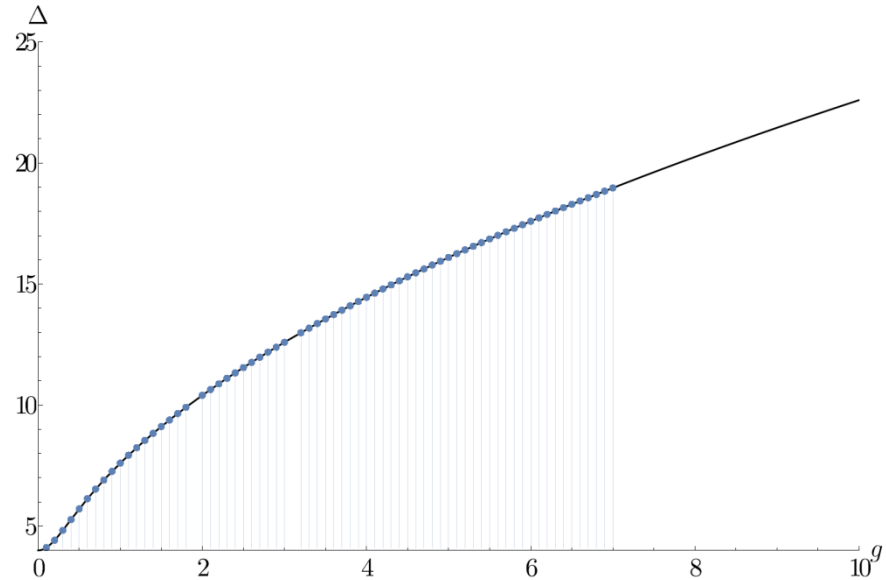
Convincing numerical agreement!

Numerical results for the Konishi

$\mathrm{Tr}\,(\Phi_I \Phi_I)$

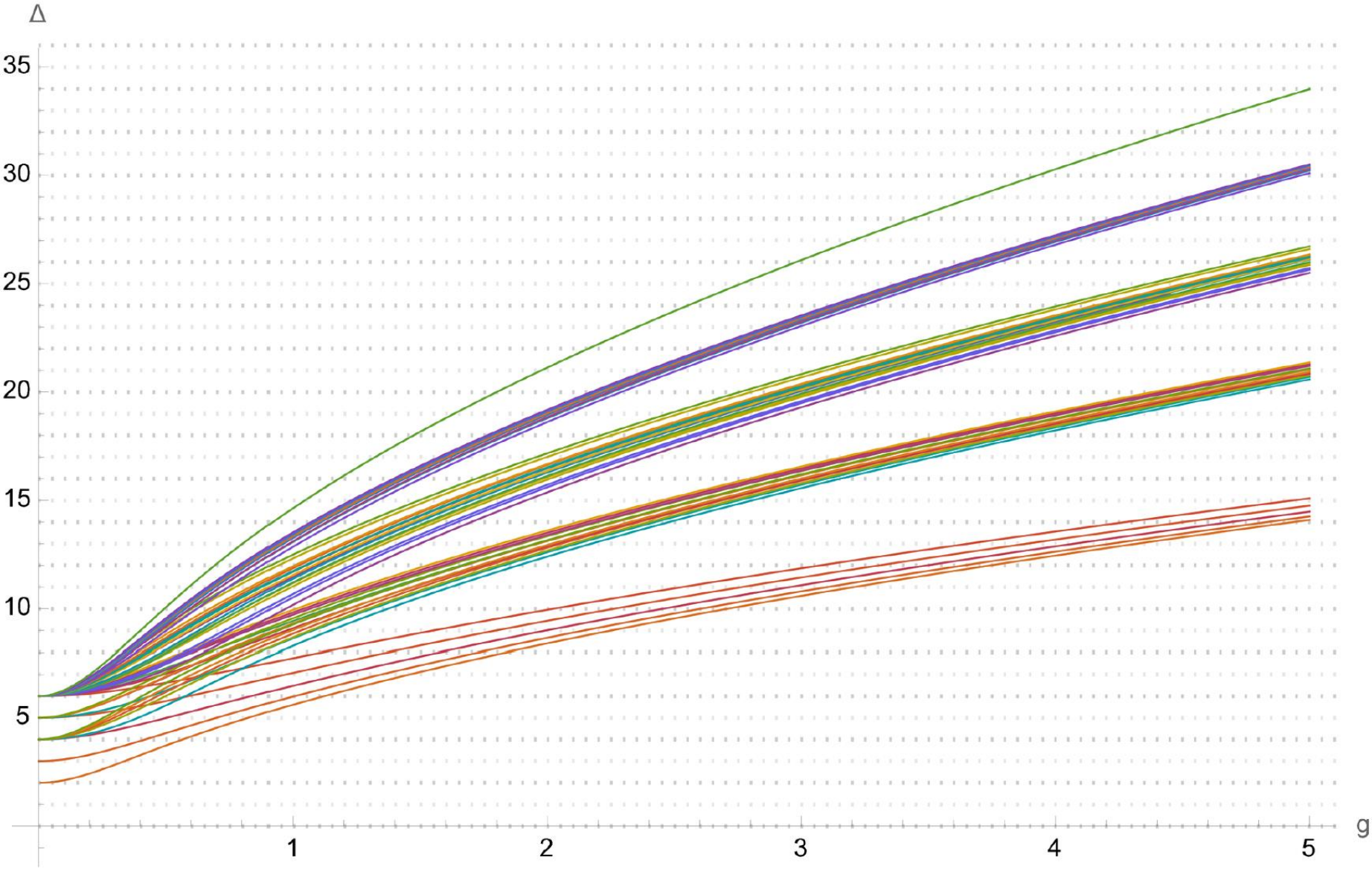
S=L=2 case

- Good agreement!



n	$\Delta^{(n)}_{exact}$	$\Delta^{(n)}_{fitted}$	$\delta_{rel}\Delta^{(n)}$
0	2.0	1.999999999999898	$5.0 \cdot 10^{-14}$
1	2.0	1.999999999995831	$2.8 \cdot 10^{-12}$
2	-3.106170709478783	-3.106170709557684	$2.5 \cdot 10^{-11}$
3	15.48929958253284	15.48929957822780	$2.8 \cdot 10^{-10}$
4	-	-91.97602372540774	$8.2 \cdot 10^{-9}$
5	-	758.5146133674111	$1.1 \cdot 10^{-6}$

Numerical results



Strong coupling analysis of data

- Expectations from string theory, for states with $[\ell_1\ \ell_2\ q_1\ p\ q_2] \sim 1$

Gubser,Klebanov, Polyakov '98

$$\Delta \simeq 2\sqrt{\delta}\,\lambda^{1/4}$$

δ- string mass level (integer)

- General expectation:

$$\Delta = \Delta_{\mathtt{reg}} + \Delta_{\mathtt{const}}$$

$$\Delta_{\mathtt{reg}} = (\delta\sqrt{\lambda})^{1/2}\left(2 + \sum_{n=1}^{\infty}\frac{d_n}{(\delta\sqrt{\lambda})^n}\right)$$

- We fitted the data according to this formula

Determination of $\Delta_{\mathtt{const}}$

- Assuming that the series of the quadratic Casimir:

$$J^2 = \frac{1}{2}(\Delta+2)^2 - 2 + \frac{1}{4}\ell_1(\ell_1+2) + \frac{1}{4}\ell_2(\ell_2+2) - \frac{1}{4}q_1(q_1+2) - \frac{1}{4}q_2(q_2+2) - \frac{1}{8}(2p+q_1+q_2)^2 - (2p+q_1+q_2)$$

- runs in powers of $1/\sqrt{\lambda}$, implies:

$$\Delta_{\mathtt{const}} = -2$$

(state independent)

- which is supported by all of our fitted values.

Determining d_1 - Kaluza-Klein towers

- At strong coupling string-states can be characterized by flat space limit quantum numbers

- $AdS_5 \times S^5 \longrightarrow SO(9) \longrightarrow SO(4) \times SO(5)$

- To each such state a Kaluza-Klein tower of SO(6) representations can be assigned :

Bianchi et al. '03

$$[\underbrace{\ell_1\ \ell_2}_{SO(4)};\ \underbrace{m\ n}_{SO(5)}] = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=m-r}^{\infty}[\ell_1\,,\ \ell_2\,,\ r+n-s\,,\ p\,,\ r+s]$$

$$+\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\sum_{p=m-r-1}^{\infty}[\ell_1\,,\ \ell_2\,,\ \underbrace{r+n-s\,,\ p\,,\ r+s+1}_{SO(6)}]$$

- Each SYM state can be associated to a KK-tower

Determining d_1

- Kaluza-Klein tower structure at δ=fixed is known:

δ=1: $[0\ 0;\ 0\ 0]$

δ=2: $2\,[0\ 0;\ 0\ 0] + [0\ 0;\ 2\ 0] + [0\ 0;\ 0\ 2] + 2\,[1\ 1;\ 1\ 0] + [2\ 2;\ 0\ 0] + [2\ 0;\ 0\ 0] + [0\ 2;\ 0\ 0]$

etc.

- The strong coupling behaviour of the Casimir:

$$J^2 \simeq 2\,\delta\sqrt{\lambda} + j_1$$

- Numerical data suggests, that j_1 is constant within a KK-tower
- This implies the following formula for the next coefficient d_1

$$d_1 = \frac{p^2}{4} + \frac{p}{4}\,(q_1 + q_2 + 4) + \frac{1}{16}\Big[16 - 2\,\ell_1\,(\ell_1 + 2) - 2\,\ell_2\,(\ell_2 + 2) + 3\,q_1\,(q_1 + 4) + 3\,q_2\,(q_2 + 4) + 2\,q_1\,q_2\Big] + \frac{j_1}{2}$$

Restricting some OPE coefficients

- The previous knowledge on strong coupling behaviour of Δs, allows one to constrain or in some cases to determine the leading strong coupling coefficients of some structure constants

Alday, Hansen '22

- Consider the 4-pt function of: (graviton scattering amplitude)

$$\mathcal{O}_2(\vec{x},\, Y) \equiv \mathrm{Tr}\, \Phi_I(\vec{x})\, \Phi_J(\vec{x})\, Y^I\, Y^J$$

20' ½-BPS

Φ_I is a fundamental real scalar

Y^I is a polarisation nullvector

- The 4-pt function is a function of cross ratios:

$$\langle \mathcal{O}_2\,(\vec{x}_1, Y_1)\, \mathcal{O}_2\,(\vec{x}_2, Y_2)\, \mathcal{O}_2\,(\vec{x}_3, Y_3)\, \mathcal{O}_2\,(\vec{x}_4, Y_4)\rangle = \frac{Y_{1\,2}^2 Y_{3\,4}^2}{x_{1\,2}^4 x_{3\,4}^4} \mathcal{S}(U, V; \sigma, \tau)$$

$$U \equiv \frac{x_{1\,2}^2 x_{3\,4}^2}{x_{1\,3}^2 x_{2\,4}^2} \equiv z\,\bar{z}\,, \quad V \equiv \frac{x_{1\,4}^2 x_{2\,3}^2}{x_{1\,3}^2 x_{2\,4}^2} \equiv (1-z)\,(1-\bar{z})$$

conformal cross ratios

$$x_{i\,j} \equiv x_i - x_j$$

$$\sigma \equiv \frac{Y_{1\,3}.\, Y_{2\,4}}{Y_{1\,2}.\, Y_{3\,4}}\,, \quad \tau \equiv \frac{Y_{1\,4}.\, Y_{2\,3}}{Y_{1\,2}.\, Y_{3\,4}}$$

R-symmetry cross ratios

$$Y_{I\,J} \equiv Y_I \cdot Y_J$$

Restricting the OPE coefficients

- Superconformal symmetry imples:

$$\mathcal{S}(U,V;\sigma,\tau) = \underbrace{\mathcal{S}_{\mathtt{free}}(U,V;\sigma,\tau)}_{\text{explicitly known}} + \underbrace{\Theta(U,V;\sigma,\tau)}_{\text{explicitly known}}\,\mathcal{T}(U,V)$$

Conformal crossing relation:

$$\mathcal{T}(U,V) = \mathcal{T}(1/U, V/U) = 1/V^2\,\mathcal{T}(U/V, 1/V)$$

Expansion in terms of conformal blocks: Alday, Hansen Silva '22

$$\mathcal{T}(U,V) = U^{-2}\sum_{T,\ell} \underbrace{C^2_{T,\ell}}_{\text{OPE coeff.}} G_{T+4,\ell}(U,V)$$

Twist: $T \equiv \Delta - \ell$

Contributing states: $[\ell\ \ell\ 0\ 0\ 0]$ $\qquad \ell = 0, 2, 4 \ldots$

Conformal blocks: Dolan, Osborne '04

$$G_{T,\ell}(U,V) = \frac{z\,\bar{z}}{z-\bar{z}}\left(k_{T+2\,\ell}(z)k_{T-2}(\bar{z}) - k_{T+2\,\ell}(\bar{z})k_{T-2}(z)\right)$$

$$k_h(z) \equiv z^{\frac{h}{2}}\,{}_2F_1(h/2, h/2, h, z)$$

Restricting the OPE coefficients

- Strong coupling structure of OPE coefficients: Alday, Hansen Silva '22

$$C^2(\lambda) = \frac{\pi^3}{2^{12}} \frac{2^{-2\,T(\lambda)} T(\lambda)^6}{\sin^2\left(\frac{\pi\,T(\lambda)}{2}\right)} \frac{1}{2^{2\,\ell}(\ell+1)} f(\lambda)$$

- f has the usual expansion:

$$f(\lambda) \simeq f_0 + \frac{f_1}{\lambda^{1/4}} + \frac{f_2}{\lambda^{1/2}}$$

The twist at strong coupling:

$$T(\lambda) \simeq 2\sqrt{\delta}\,\lambda^{\frac{1}{4}} - 2 - \ell + \frac{T_1}{\lambda^{1/4}} \qquad T_1 \equiv \frac{d_1}{\sqrt{\delta}}$$

The values of δ and T_1 are known from our numerical work for the lowest lying states, which is necessary to restrict the values for the coefficients $f_{0,1,2}$

Restricting the OPE coefficients

- Comparison of strong coupling string theory results to conformal block expansion led to sums rules along "Regge-trajectories" defined by the numbers:

Alday, Hansen Silva '22

$$t \equiv \delta - \ell/2$$

The averages: $\langle f_0 \rangle$, $\langle f_2 \rangle$, $\langle f_1 \rangle$, $\langle T_1 f_0 \rangle$
are available on various Regge-trajectories

- On the 1st Regge-trajectory: t=1, upto $\delta \leq 7$, there is only one state, so average gives the exact values for f_0 , f_1 , f_2 for these special states

On the 2nd Regge-trajectory t=2 with δ=2, there are only 2 states. Thus our knowledge on T_1 allows us to determine, the leading coefficients for these states.

Restricting the OPE coefficients

- The 2 states on the 2nd Regge-trajectory with $\delta=2$ have the quantum numbers:

$[0,0,0,0,0]$ $\Delta_0=L=4$ $B=0$

They are degenerate at $g=0$, only quantum corrections lift this degeneracy:

$$\Delta=\Delta_0+g^2\,(13\pm\sqrt{41})+\ldots$$

"-" 3rd
"+" 4th
states in our database

From fitting the numerical data at strong coupling:

$\delta=2$ $T_{1;\,3}=\sqrt{2}\,,$ $T_{1;\,4}=4\sqrt{2}$

$$T(\lambda)\simeq 2\sqrt{\delta}\,\lambda^{\frac{1}{4}}-2-\ell+\frac{T_1}{\lambda^{1/4}}$$

$$T\equiv\Delta-\ell$$

Restricting the OPE coefficients

- The average formulas on the 2nd Regge-trajectory:

$$\langle f_0 \rangle = \frac{r_1(\delta)}{3}\left(2\delta^2 + 3\delta - 8\right)$$

Alday, Hansen Silva '22

$$\langle f_0\, T_1 \rangle = \frac{r_1(\delta)}{18\sqrt{\delta}}\left(18\delta^4 + 25\delta^3 - 57\delta^2 + 50\delta - 72\right)$$

where:

$$r_n(\delta) = \frac{4^{2-2\delta}\,\delta^{2\delta-2n-1}(2\delta - 2n - 1)}{\Gamma(\delta)\Gamma\left(\delta - \left\lfloor \frac{n}{2} \right\rfloor\right)}$$

- Applying to our case with $\delta=2$:

$$\langle f_0 \rangle = f_{0;\,3} + f_{0;\,4} = \frac{1}{4}$$

$$\langle f_0\, T_1 \rangle = f_{0;\,3}\, T_{1;\,3} + f_{0;\,4}\, T_{1;\,4} = \sqrt{2}$$

Solution:

$$\boxed{f_{0;\,3} = 0\,, \qquad f_{0;\,4} = \frac{1}{4}}$$

Restricting OPE coefficients

- On other Regge-trajectories there are more operators (unknows), than the number of equations for averages. Thus, we cannot make exact determination of the coeffcients $f_{0,1,2}$.
- However, using their positivity: $f_{0,1,2} \geq 0$, inequalities giving lower and upper bonds can be derived.

In a recent paper: J. Julius, N. Sokolova '23

Assuming: j1=const. in a KK-tower, could show that:

-f_0 is the same for all states within a KK-tower

-they could determine the values for f_0

Summary

- We solved numerically the QSC equations with high precision for all the 219 states with $\Delta_0 \leq 6$ in a wide range of the 't Hooft coupling.

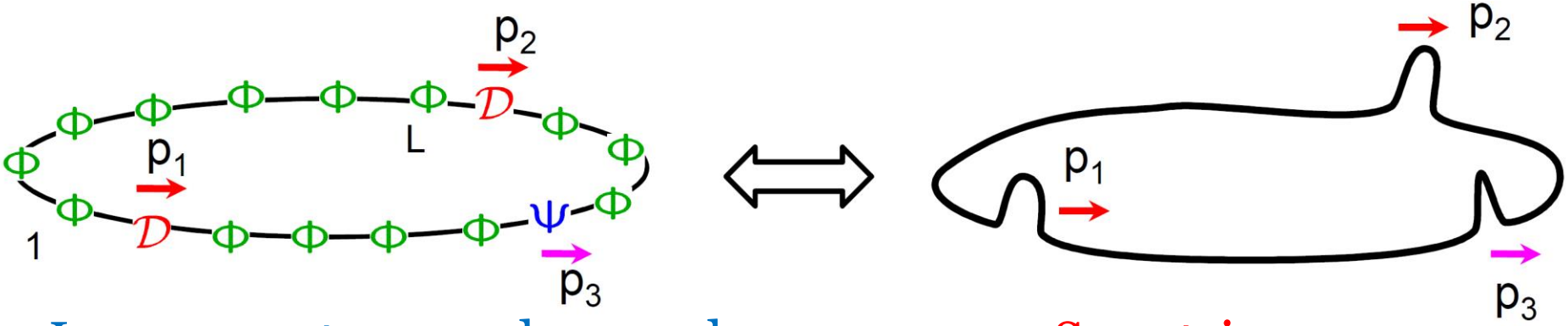
- We fitted (at least) the first 3 coefficients of the strong coupling series of the Δs

- With these data, we could associate the states to specific Kaluza-Klein towers

- We could restrict some OPE coefficients of our states with two rank-2 symmetric traceless ½-BPS operators.

Spin-chain description

- Planar limit: single trace → spin chain description
- SUSY protected BPS states: Tr(ΦΦΦ…Φ)
- Vacuum: Tr(ΦΦΦ…Φ) ↔ string vacuum



Long operators ↔ large volume: S-matrix

$$psu(2,2|4) \Longrightarrow su(2|2) \times su(2|2)$$

$$S_{PSU(2,2|4)}(p_1,p_2) = S_0^2(p_1,p_2)\ S_{SU(2|2)}(p_1,p_2) \times S_{SU(2|2)}(p_1,p_2)$$