Constraints on the Dirac spectrum from chiral symmetry restoration in high-temperature QCD

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QCD at Finite Temperature

Analytic crossover in the range $T \simeq 145 - 165$ MeV from hadronic phase to quark-gluon-plasma phase [BW collaboration (2010)]

Still many open questions:

- What is the microscopic mechanism behind the transition?
- What is the relation between χSR and deconfinement?
- What is the fate of the anomalous $\mathrm{U}(1)_A$ symmetry in the plasma?

Spectrum/eigenvectors of the Dirac operator encode quark dynamics, good place to look for clues: how does χSR reflect on them?

Lightest u, d quarks $m_{u,d} \lll \Lambda_{\mathrm{QCD}} \approx 1\ \mathrm{GeV}$
$\Rightarrow$ study $N_f = 2$ **chiral limit**, $m_{u,d} \to 0$

- good approximation of QCD
- symmetric starting point for systematic study

Finite-Temperature Gauge Theory

Partition function at finite T (imaginary time/Euclidean formulation)

$$Z = \int [DA] \int [D\psi D\bar{\psi}]\, e^{-S_G[A]-S_F[\psi,\bar{\psi},A]}$$

$$= \int [DA]\ \left[\det(\not{D}[A]+m)\right]^2 \overbrace{\textstyle\prod_h \det(\not{D}[A]+m_h)e^{-S_G[A]}}^{e^{-S_{\rm eff}[A]}}$$

$$\boxed{\text{Dirac operator: } \not{D}[A] = (\partial_\mu + igA^a_\mu t^a)\gamma_\mu}$$

- gauge fields A^a_μ, fermion fields $\psi, \bar{\psi}$
- two light ($\to$massless) flavours, possibly other massive ones
- compact time direction of size $\frac{1}{T}$, finite spatial volume V_3, $\mathrm{V}_4 = \frac{\mathrm{V}_3}{T}$
- periodic/antiperiodic temporal b.c. for gauge/fermion fields
- first $\mathrm{V}_3 \to \infty$, then $m \to 0$

$$S_G = \frac{1}{4}\int_{\mathrm{V}_4} d^4x\, F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)$$

$$S_F = \int_{\mathrm{V}_4} d^4x\ \bar{\psi}(x)(\not{D}[A]+m)\psi(x)$$

$$\int_{\mathrm{V}_4} d^4x = \int_0^{\frac{1}{T}} dt \int_{\mathrm{V}_3} d^3x$$

Dirac Spectrum

$\not{D}$: anti-Hermitean operator, chiral symmetry $\{\gamma_5, \not{D}\} = 0$

$$\not{D}[A]\psi_n[A] = i\lambda_n[A]\psi_n[A] \qquad \lambda_n[A] \in \mathbb{R}$$

Purely imaginary spectrum, symmetric about the origin

Spectral density and connected two-point function of nonzero modes:

$$\rho(\lambda) \equiv \lim_{\mathrm{V}_4\to\infty} \langle \rho_A(\lambda) \rangle$$

$$\rho_c^{(2)}(\lambda, \lambda') \equiv \lim_{\mathrm{V}_4\to\infty} \Big(\mathrm{V}_4 \left\langle \rho_A(\lambda)\rho_A(\lambda') \right\rangle_c - \delta(\lambda - \lambda') \left\langle \rho_A(\lambda) \right\rangle \Big)$$

$$\rho_A(\lambda) \equiv \frac{1}{\mathrm{V}_4} \sum_{n, \lambda_n \neq 0} \delta(\lambda - \lambda_n[A]) = \rho_A(-\lambda)$$

$\langle AB \rangle_c \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle$

Chiral zero modes

$$\not{D}[A]\psi_0^{\pm}[A] = 0 \qquad \gamma_5 \psi_0^{\pm}[A] = \pm\psi_0^{\pm}[A]$$

$N_{\pm}$: n° of zero modes of chirality ± 1, $N_0 = N_+ + N_-$

QCD in the Chiral Limit

At $m_{u,d} = 0$ exact chiral symmetry

$$\mathrm{U}(2)_L \times \mathrm{U}(2)_R \sim \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$$

Flavour non-singlet symmetries:

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \sim \mathrm{SU}(2)_V \times \mathrm{SU}(2)_A$$

$$\psi \to \psi_{\mathcal{U}} = \begin{cases} e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\psi & \mathrm{SU}(2)_V \\ e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}\gamma_5}\psi & \mathrm{SU}(2)_A \end{cases} \qquad \bar{\psi} \to \bar{\psi}_{\mathcal{U}} = \begin{cases} \bar{\psi}e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} & \mathrm{SU}(2)_V \\ \bar{\psi}e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}\gamma_5} & \mathrm{SU}(2)_A \end{cases}$$

$\mathrm{SU}(2)_V$ cannot break spontaneously [Vafa, Witten (1984), MG (2023)]

$\mathrm{SU}(2)_A$ spontaneously broken, 3 Goldstone bosons ($\Rightarrow$ light pions @m_{phys})

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \to \mathrm{SU}(2)_V$$

Symmetry-restoring phase transition at $T_c \approx 132\,\mathrm{MeV}$ [Ding *et al.* (2019)]

Flavour-singlet symmetries:

- $\mathrm{U}(1)_V \sim$ baryon number symmetry, unbroken
- $\mathrm{U}(1)_A \sim$ anomalous symmetry ($\Rightarrow$ heavy η' @m_{phys})

χSB and the Dirac Spectrum

Order parameter for $\mathrm{SU}(2)_A$: chiral condensate

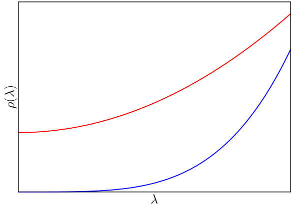
$$\langle\bar{\psi}_f(x)\psi_{f'}(x)\rangle = \Sigma\delta_{ff'} = \frac{1}{N_f}\langle\bar{\psi}(x)\psi(x)\rangle\delta_{ff'} = \frac{\langle S\rangle}{\mathrm{V}_4}\frac{\delta_{ff'}}{N_f} \qquad f, f' \in \{u, d\}$$

$\Sigma \propto$ density of near-zero Dirac modes [Banks, Casher (1980)]

$$-\Sigma = \int_0^\infty d\lambda\,\frac{2m\rho(\lambda)}{\lambda^2+m^2} \underset{m\to 0}{\to} \pi\rho(0^+)$$

What happens to the spectral density at finite T? At small m

@low T:	$\Sigma \neq 0$,	expect $\rho(0^+) \neq 0$
@high T:	$\Sigma = 0$,	expect $\rho(0^+) = 0$



What happens to the peak in the chiral limit? How is $\mathrm{SU}(2)_A$ restored?

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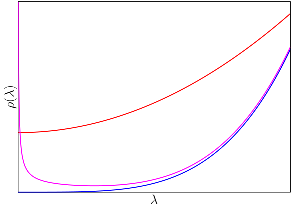
...instead singular peak [Edwards *et al.* (1992), Alexandru, Horváth (2015), Meng *et al.* (2023)]



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Susceptibilities and the Dirac Spectrum

Susceptibilities = integral of connected correlators of densities ($/\mathrm{V}_4$)

$$S = \bar{\psi}\psi \; \left(= \int_{\mathrm{V}_4} d^4x \, \bar{\psi}(x)\psi(x)\right) \qquad \vec{P} = \bar{\psi}\vec{\sigma}\gamma_5\psi \; \left(= \textstyle\sum_{f,f'} \bar{\psi}_{f'}\vec{\sigma}_{f'f}\gamma_5\psi_f\right)$$

$$P = \bar{\psi}\gamma_5\psi \qquad \vec{S} = \bar{\psi}\vec{\sigma}\psi$$

Expressed through eigenvalues only (no eigenvectors)

$$\frac{\langle P_a P_b \rangle_c}{\mathrm{V}_4} = -\delta_{ab} \int_0^\infty d\lambda \, \rho(\lambda) \frac{4}{\lambda^2 + m^2}$$

If $\mathrm{SU}(2)_A$ is restored in the chiral limit

$$\lim_{m\to 0} \tfrac{1}{\mathrm{V}_4} \langle S \rangle_c = 0 \,, \qquad \lim_{m\to 0} \tfrac{1}{\mathrm{V}_4} \langle SS \rangle_c = \lim_{m\to 0} \tfrac{1}{\mathrm{V}_4} \langle P_a P_a \rangle_c \,, \qquad \ldots$$

$\Rightarrow$ constraints on ρ and n-point connected eigenvalue correlators

Anomalous $\mathrm{U}(1)_A$ and the Dirac Spectrum

Anomalous divergence of singlet axial current related to topology

$$\partial_\mu \langle \left[\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)\right] \mathcal{O}\rangle - 2m\langle \left[\bar{\psi}(x)\gamma_5\psi(x)\right] \mathcal{O}\rangle - 2N_f \langle q(x)\mathcal{O}\rangle = i\langle \delta_A \mathcal{O}(x)\rangle$$

Anomalous Ward-Takahashi identity

$$q(x) = \tfrac{1}{64\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x) \qquad Q = \int d^4x\, q(x) \in \mathbb{Z}$$

Index theorem relates topological charge and Dirac zero modes

$$Q = N_+ - N_-$$

	$0 = \langle Q^{2k+1}\rangle$	(by CP symmetry)
topological susceptibility	$\chi_t = \lim_{\mathrm{V}_4\to\infty} \frac{\langle Q^2\rangle}{\mathrm{V}_4}$	$\neq 0$
density of zero modes	$n_0 = \lim_{\mathrm{V}_4\to\infty} \frac{\langle N_0\rangle}{\mathrm{V}_4} = 0$	(since $\langle N_+ N_-\rangle = 0$)

Anomalous $\mathrm{U}(1)_A$ in the Chiral Limit

Using integrated WT identities

$$\chi_t = -\frac{m}{2}\Sigma + \frac{m^2}{4}\frac{1}{\mathrm{V}_4}\langle PP\rangle \underset{m\to 0}{\to} 0$$

but topology can still lead to $\mathrm{U}(1)_A$ breaking visible in susceptibilities

$$-\tfrac{1}{\mathrm{V}_4}\left(\langle SS\rangle_c + \langle PP\rangle_c\right) = \frac{2}{m^2}\frac{\langle N_0^2\rangle}{\mathrm{V}_4} + \int_0^\Lambda d\lambda\,\rho(\lambda)\frac{4m^2}{(\lambda^2+m^2)^2}$$

χ_t strongly suppressed at high T at $m \neq 0$, "double suppression" as $m \to 0$: Is $\mathrm{U}(1)_A$ effectively restored in the chiral limit above T_c?

Universality class of the chiral transition depends on the fate of $\mathrm{U}(1)_A$

[Pisarski, Wilczek (1984), Pelissetto, Vicari (2013)]

Do constraints on the spectrum from $\mathrm{SU}(2)_A$ restoration tell us something about effective $\mathrm{U}(1)_A$ breaking/restoration as well?

Constraints on Dirac spectrum and topology from χSR

- [Cohen (1996)]: $\boxed{\mathrm{SU}(2)_A \text{ restored} \Rightarrow \mathrm{U}(1)_A \text{ restored}}$

  Assumes observables (including ρ) analytic in m^2, topology suppressed

- [Evans, Hsu, Schwetz (1996)], $\boxed{\mathrm{SU}(2)_A \text{ restored} \not\Rightarrow \mathrm{U}(1)_A \text{ restored}}$

  [Lee, Hatsuda (1996)]: $\boxed{\mathrm{SU}(2)_A \text{ restored} \not\Rightarrow \mathrm{U}(1)_A \text{ restored}}$

  Assume interchangeability of thermodynamic and chiral limits

- [Aoki, Fukaya, Taniguchi (2012)]: $\boxed{\mathrm{SU}(2)_A \text{ restored} \Rightarrow \mathrm{U}(1)_A \text{ restored}}$

  Assumes ρ analytic in m^2, power series in λ at 0 (can be relaxed to $\rho \sim \lambda^\alpha$, $\alpha > 0$) plus strong technical assumption on ρ_A; shows topology suppressed

- [Kanazawa, Yamamoto (2015)]: $\boxed{\mathrm{SU}(2)_A \text{ restored} \Rightarrow \mathrm{U}(1)_A \text{ restored}}$

  Drop strong technical assumption of Aoki

- [Azcoiti (2023)]: $\boxed{\mathrm{SU}(2)_A \text{ restored} \Rightarrow \mathrm{U}(1)_A \text{ restored}}$

  Assuming both m^2-analyticity and interchangeability of limits, $\mathrm{U}(1)_A$ can be broken only if $\rho \sim m^2 f_A \delta(\lambda)$ (unlikely) – is $\mathrm{SU}(2)_A$ fully restored then?

Which assumptions can we/must we keep?

Chiral Symmetry Restoration for S/PS Susceptibilities - I

Symmetry restoration $\Rightarrow$ local correlators related by $\mathrm{SU}(2)_A$ are equal

No Goldstones in the restored phase, finite correlation length
$\Rightarrow$ susceptibilities related by $\mathrm{SU}(2)_A$ transformations are equal

$$e^{-\mathrm{V}_4 \mathcal{W}(J;m)} = \int [DA]\, e^{-S_{\mathrm{eff}}[A]} \int [D\psi D\bar{\psi}]\, e^{-\bar{\psi}(\not{D}[A]+m)\psi - K(\psi,\bar{\psi},A;J)}$$

$$K(\psi,\bar{\psi},A;J) = J_S S + i\vec{J}_P \cdot \vec{P} + iJ_P P - \vec{J}_S \cdot \vec{S}$$

Susceptibilities $\propto$ derivatives of generating function $\mathcal{W}$ wrt sources
To get generating function of $\mathrm{SU}(2)_A$-transformed susceptibilities, replace

$$K(\psi,\bar{\psi},A;V,W) \to K(\psi_{\mathcal{U}},\bar{\psi}_{\mathcal{U}},A;V,W) = K(\psi,\bar{\psi},A;R_{\mathcal{U}}V,R_{\mathcal{U}}W)$$

$$\mathcal{U} \in \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \leftrightarrow R_{\mathcal{U}} \in \mathrm{SO}(4)$$

$O_{V,W} \to R_{\mathcal{U}}^T O_{V,W}$

Chiral symmetry restoration in the chiral limit:

$$\lim_{m\to 0} \mathcal{W}(V,W;m) = \lim_{m\to 0} \mathcal{W}(R_{\mathcal{U}}V,R_{\mathcal{U}}W;m)$$

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$$K(\psi,\bar{\psi},A;J) = (J_S,\vec{J}_P)\cdot(S,i\vec{P}) + (J_P,\vec{J}_S)\cdot(iP,-\vec{S}) = V\cdot O_V + W\cdot O_W$$

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$$\lim_{m\to 0} \mathcal{W}(V,W;m) = \lim_{m\to 0} \mathcal{W}(R_{\mathcal{U}}V,R_{\mathcal{U}}W;m)$$

Restrictions on $\mathcal{W}$

1. $\mathcal{W}(V, W; m)$ depends on J_S and m only through $J_S + m$

$$\mathcal{W}(V, W; m) = \mathcal{W}(\tilde{V}, W; 0) \quad \text{with} \quad \tilde{V} = V + (m, \vec{0})$$

2. $\mathcal{W}(V, W; 0)$ chirally symmetric, $\mathcal{W}(V, W; 0) = \hat{\mathcal{W}}(V^2, W^2, 2V \cdot W)$

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3. $\mathcal{W}$ function of J_S, J_P and $\mathrm{SO}(4)$ invariants $Y = (V^2, W^2, 2V \cdot W)$

$$\lim_{m\to 0} \bar{\mathcal{W}}(J_S, J_P, Y; m) = \lim_{m\to 0} \bar{\mathcal{W}}(0, 0, Y; m) \qquad \text{(restoration condition)}$$

$$\partial_{J_S} \left(\prod_i \partial_{Y_i}^{n_i}\right) \bar{\mathcal{W}}(J_S, J_P, Y; m) = 2m\partial_u^{n_1+1}\partial_w^{n_2}\partial_{\tilde{u}}^{n_3}\hat{\mathcal{W}}(m^2 + u, w, \tilde{u}) \underset{m\to 0}{\to} 0$$

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Chiral Symmetry Restoration for S/PS Susceptibilities - II

Generating function of scalar/pseudoscalar susceptibilities

$$\mathcal{W}(V, W; m) = \hat{\mathcal{W}}(m^2 + u, w, \tilde{u})$$

Formal power series in $u = 2mJ_S + V^2$, $w = W^2$, $\tilde{u} = 2(mJ_P + V \cdot W)$

$$\mathcal{W}(V, W; m) = \sum_n \frac{1}{n_1!n_2!n_3!} \mathcal{A}_{n_1,n_2,n_3}(m^2) u^{n_1} w^{n_2} \tilde{u}^{n_3}$$
$$\mathcal{A}_n(m^2) \equiv \partial_u^{n_1} \partial_w^{n_2} \partial_{\tilde{u}}^{n_3} \hat{\mathcal{W}}(m^2 + u, w, \tilde{u})|_0$$

$$\boxed{\mathrm{SU}(2)_A \text{ restored} \Leftrightarrow \mathcal{A}_n(m^2) = \sum_{k=0}^{\infty} m^{2k} a_n^{(k)},\ |a_n^{(k)}| < \infty}$$

Up to terms vanishing faster than any power
Possibly zero radius of convergence

$\boxed{\Longleftarrow}$ $\mathcal{A}_n(0)$ finite $\Rightarrow u, w, \tilde{u} \underset{m\to 0}{\to} V^2, W^2, 2V \cdot W$

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$\Rightarrow \mathcal{A}_n(0)$ finite $\qquad \Rightarrow (\partial_{m^2})^k \mathcal{A}_n(0) = \mathcal{A}_{n_1+k,n_2,n_3}(0)$ finite

Chiral Symmetry Restoration for S/PS Susceptibilities - II

Generating function of scalar/pseudoscalar susceptibilities

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Chiral Symmetry Restoration for S/PS Susceptibilities - III

Scalar and pseudoscalar susceptibilities are power series in m^2
if chiral symmetry is restored (for $N_f \geq 2$, not guaranteed for $N_f = 1$)

Naturally extends to correlation functions involving local functionals of gauge fields, and corresponding integrated connected correlation functions

$$K[\psi, \bar{\psi}, A; V, W] \to K[\psi, \bar{\psi}, A; V, W] + \underbrace{\mathcal{J} \int d^4x\, \mathcal{G}[A(x)]}_{\text{unaffected by chiral transformations}}$$

Likely to extend to correlation functions involving also nonlocal functionals

$$\rho(\lambda; m) = \sum_{k=0}^{\infty} m^{2k} \rho_k(\lambda)$$

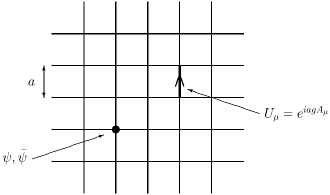
For ρ, also argument using local operators in PQ theory

What are the consequences for the spectrum?

Gauge Theories on the Lattice

Path integral ill-defined, needs regularisation

Lattice approach: Euclidean continuum replaced with discrete, finite lattice



$$Z_{\rm lat} = \int [dU]\, \det(D_{\rm lat}[U] + m)^2 e^{-S_{\rm eff}^{\rm lat}[U]}$$

Discretising $\not{D}$ problematic for $\mathrm{SU}(2)_A \Rightarrow$ Ginsparg-Wilson fermions

$$\{D_{\rm GW}, \gamma_5\} = 2D_{\rm GW} R \gamma_5 D_{\rm GW} \qquad (R \text{ local})$$ [Ginsparg, Wilson (1982)]

Possess exact lattice chiral symmetry [Lüscher (1998)]

Exact chiral multiplets

$$S = \bar{\psi}(1 - D_{\rm GW} R)\psi \qquad P = \bar{\psi}(1 - D_{\rm GW} R)\gamma_5\psi$$
$$\vec{P} = \bar{\psi}(1 - D_{\rm GW} R)\vec{\sigma}\gamma_5\psi \qquad \vec{S} = \bar{\psi}(1 - D_{\rm GW} R)\vec{\sigma}\psi$$

$\Rightarrow$ everything works as in the continuum – except it exists. . .

First-order Constraints - I

$\mathcal{W}$ function of $\bar{u} = \tilde{u}^2$ thanks to CP symmetry

$$\mathcal{W} = \mathcal{C}_0 + u\mathcal{C}_u + w\mathcal{C}_w + \bar{u}\mathcal{C}_{\bar{u}}$$
$$+ \frac{1}{2}\left(u^2\mathcal{C}_{uu} + 2uw\mathcal{C}_{uw} + w^2\mathcal{C}_{ww} + 2u\bar{u}\mathcal{C}_{u\bar{u}} + 2w\bar{u}\mathcal{C}_{w\bar{u}} + \bar{u}^2\mathcal{C}_{\bar{u}\bar{u}}\right) + \dots$$

After some algebra. . .

▸ details

$$\mathcal{C}_u = \frac{\chi_\pi}{2} = \frac{n_0}{m^2} + 2I^{(1)}[f] \qquad \Rightarrow \lim_{m\to 0} \frac{\chi_\pi}{4} = \lim_{m\to 0} \int f\rho < \infty$$

$$\mathcal{C}_w = \frac{\chi_\delta}{2} = -\frac{n_0}{m^2} + 2I^{(1)}[\tilde{f}] \qquad \Rightarrow \left| \lim_{m\to 0} \frac{\chi_\delta}{4} \right| \leq \lim_{m\to 0} \frac{\chi_\pi}{4}$$

$$\mathcal{C}_{\bar{u}} = \frac{1}{m^2}\left(\frac{n_0 - \chi_t}{2m^2} + m^2 I^{(1)}[f^2]\right) \qquad \Rightarrow \frac{\chi_t}{m^2} = 2m^2 \int f^2\rho + O(m^2)$$

$$I^{(1)}[g] \equiv \int_0^2 d\lambda\, g(\lambda)\rho(\lambda; m) \qquad \Rightarrow \lim_{m\to 0} \frac{\chi_t}{m^2} = \lim_{m\to 0} 2m^2 \int f^2\rho \leq \frac{\chi_\pi}{2}$$

$$f \equiv \frac{1-\lambda^2/4}{\lambda^2+m^2(1-\lambda^2/4)} \qquad \tilde{f} \equiv f - 2m^2 f^2 \qquad (\text{use } n_0 = 0,\, m^2 f^2 \leq f)$$

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First-order Constraints - II

Restoration conditions $\lim_{m\to 0} \mathcal{C}_{u,w,\bar{u}} < \infty$ boil down to

$$\lim_{m\to 0} \frac{\chi_\pi}{4} = \lim_{m\to 0} \int_0^2 d\lambda\, f(\lambda, m)\rho(\lambda; m) < \infty$$

$$\Rightarrow \lim_{m\to 0} \int_0^\delta d\lambda\, \frac{\rho(\lambda; m)}{\lambda^2 + m^2} < \infty$$

$$\lim_{m\to 0} \frac{\chi_t}{m^2} = \lim_{m\to 0} 2m^2 \int_0^2 d\lambda\, f(\lambda, m)^2 \rho(\lambda; m)$$

$$= \lim_{m\to 0} 2m^2 \int_0^\delta d\lambda\, \frac{\rho(\lambda; m)}{(\lambda^2 + m^2)^2} \qquad (\text{any } \delta > 0)$$

$\mathrm{U}(1)_A$ order parameter:

$$\Delta \equiv \lim_{m\to 0} \frac{\chi_\pi - \chi_\delta}{4} = \lim_{m\to 0} 2m^2 \int_0^2 d\lambda\, f(\lambda, m)^2 \rho(\lambda; m) = \lim_{m\to 0} \frac{\chi_t}{m^2}$$

Fate of $\mathrm{U}(1)_A$ undecided, assumptions on ρ are needed

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Spectral Density - I

Assume ρ is an ordinary function (no δs)

1. If in m-independent neighbourhood of zero

$$\rho(\lambda; m) = \sum_{n=0}^{\infty} \rho_n(m)\lambda^n$$

$\Rightarrow \mathrm{SU}(2)_A$ restoration requires

$$\rho_0 = O(m) \qquad \rho_1(m) = O(1/\ln|m|)$$

$\Rightarrow \mathrm{U}(1)_A$ order parameter

$$\Delta = \frac{\pi}{2} \lim_{m\to 0} \frac{\rho_0(m)}{m}$$

$\Rightarrow$ if ρ power series in m^2 then $\Delta = 0$

[Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2015)]

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Spectral Density - II

Assume ρ is an ordinary function (no δs)

2. Numerical indications of singular peak, if ρ is a power law at small λ

$$\rho(\lambda; m) \simeq C(m)\lambda^{\alpha(m)}$$

$\Rightarrow \mathrm{SU}(2)_A$ restoration requires

$$C(m) = \frac{\cos\left(\alpha(m)\frac{\pi}{2}\right)}{(1-\alpha(0))\frac{\pi}{2}} m^{1-\alpha(0)}(c + o(1))$$

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$$\Delta = c$$

$\Rightarrow \mathrm{U}(1)_A$ breaking visible if $c \neq 0$
$\Rightarrow$ cannot be power series in m^2 except if $\alpha(m) \to -1$ as $m \to 0$

If $\alpha(m)$ and $C(m) = m^{2\frac{1+\alpha(m)}{2}}(\Delta + \hat{C}(m))$ are power series in m^2

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$\mathrm{U}(1)_A$ Breaking from Singular Peak

Ordinary function ρ compatible with $\mathrm{SU}(2)_A$ restoration, $\mathrm{U}(1)_A$ breaking, m^2 analyticity if it has singular peak tending to m^4/λ

$$\rho_{\rm sing}(\lambda; m) = m^2 \frac{\Delta}{2} \frac{\gamma m^2 + O(m^4)}{\lambda^{1-\gamma m^2 + O(m^4)}} = \frac{\chi_t}{2} \frac{\partial \lambda^{\gamma m^2}}{\partial \lambda}(1 + O(m^2))$$

- Thermodynamic and chiral limits do not commute
- Argument based on dilute instanton gas reproduces singular peak features, peak $\approx$ zero modes of isolated (anti)instantons after mixing [Kovács (2023)]
- Density of peak modes $n_{\rm peak} = \lim_{\mathrm{V}_4 \to \infty} N_{\rm peak}/\mathrm{V}_4$

$$\lim_{m\to 0} \frac{n_{\rm peak}}{m^2} = \lim_{m\to 0} \frac{2}{m^2} \int_0^2 d\lambda\, \rho_{\rm sing}(\lambda; m) = \Delta = \lim_{m\to 0} \frac{\chi_t}{m^2}$$

$\Rightarrow$ consistent with modes originating in the zero-modes of a dilute instanton/anti-instanton gas (density $= \chi_t/2$)

Second-order Constraints - I

Higher-order constraints involve higher-order eigenvalue correlators (no more constraints on ρ)

Two constraints involving $\rho_c^{(2)}$

$$\frac{b_{N_0^2} - \chi_t}{m^2} - 2m^2 \frac{\partial}{\partial m^2}\frac{n_0}{m^2} - 4m^2 I^{(2)}[f,f] = O(m^2)$$
$$I^{(2)}[\hat{f},\hat{f}] = O(m^0)$$

$$I^{(2)}[g_1,g_2] \equiv \int_0^2 d\lambda \int_0^2 d\lambda' \, g_1(\lambda) g_2(\lambda') \rho_c^{(2)}(\lambda,\lambda';m)$$
$$\hat{f} \equiv f - m^2 f^2 \qquad b_{N_0^2} \equiv \lim_{\mathrm{V}_4\to\infty} \frac{\langle N_0^2\rangle - \langle N_0\rangle^2}{\mathrm{V}_4}$$

Using $\langle N_+ N_-\rangle = 0 \Rightarrow \lim_{m\to 0} 4m^2 I^{(2)}[f,f] = -\lim_{m\to 0}\lim_{\mathrm{V}_4\to\infty} \frac{\langle N_0\rangle^2}{m^2 \mathrm{V}_4}$

Assume $\rho_c^{(2)}$ ordinary function (no δs)

Two-Point Function - I

1. If $\rho_c^{(2)}(\lambda, \lambda'; m)$ finite at the origin

$$\rho_c^{(2)}(\lambda, \lambda'; m) = A(m) + B(\lambda, \lambda'; m)$$

$$|B(\lambda, \lambda'; m)| \leq b(\lambda^2 + \lambda'^2)^{\frac{\beta}{2}} \text{ with } \beta < 1$$

$$\lim_{m\to 0} 4m^2 I^{(2)}[f, f] = \pi^2 A(0) = -\lim_{m\to 0} \lim_{\mathrm{V}_4\to\infty} \frac{\langle N_0\rangle^2}{m^2 \mathrm{V}_4}$$

$$\lim_{m\to 0} (4m)^2 I^{(2)}[\hat{f}, \hat{f}] = \pi^2 A(0) = 0$$

$\Rightarrow$ measure of $\frac{N_0}{m\sqrt{\mathrm{V}_4}}$ concentrated in zero in thermo and chiral limit

$\Rightarrow \lim_{m\to 0} \frac{\chi_t}{m^2} = 0$

No $\mathrm{U}(1)_A$-breaking topological effects if $\rho_c^{(2)}(\lambda, \lambda'; m)$ finite at zero

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if $\rho_c^{(2)}(\lambda,\lambda';m)$ finite at zero

Two-Point Function - II

2. Above T_c, low modes localised below "mobility edge", λ_c, obey Poisson statistics [MG, Kovács (2021)]

   Purely Poisson spectrum of $\mathrm{N_P V_4}$ modes [Kanazawa, Yamamoto (2015)]

$$\rho^{(2)}_{\mathrm{P}\,c}(\lambda,\lambda') = -\frac{1}{\mathrm{N_P}}\rho_{\mathrm{P}}(\lambda)\rho_{\mathrm{P}}(\lambda')$$

   $\Rightarrow$ expect $|\rho^{(2)}_c(\lambda,\lambda';m)| \le C\rho(\lambda;m)\rho(\lambda';m)$ for $\lambda,\lambda' < \lambda_c$.

   Localised modes fluctuate independently of each other

   $\Rightarrow$ expect $|\rho^{(2)}_c(\lambda,\lambda';m)| \le C'\rho(\lambda;m)$ for $\lambda < \lambda_c < \lambda'$.

$$\text{If } \lambda_c \underset{m\to 0}{\not\to} 0 \Rightarrow \lim_{m\to 0}\lim_{\mathrm{V_4}\to\infty}\frac{\langle N_0\rangle^2}{m^2\mathrm{V_4}} = -\lim_{m\to 0} 4m^2 I^{(2)}[f,f] = 0$$

$\mathrm{U}(1)_A$-breaking topological effects require

- either $\lambda_c \to 0$ as $m \to 0$
- or another $\lambda'_c < \lambda_c$ exist close to zero

Two-Point Function - II

2. Above T_c, low modes localised below "mobility edge", λ_c, obey Poisson statistics [MG, Kovács (2021)]

Purely Poisson spectrum of $\mathrm{N_P V_4}$ modes [Kanazawa, Yamamoto (2015)]

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Second-order Constraints - II

Further constraints involve topology

$$\left|\lim_{m\to 0} \frac{\partial}{\partial m^2}\frac{\chi_t}{m^2}\right| < \infty \qquad \left|\lim_{m\to 0} \mathcal{C}_{u\bar{u}}\right| < \infty$$

Imply the requests

$$\frac{\chi_t}{m^2} = \Delta + \Delta_1 m^2 + o(m^2)$$
$$\frac{n_0 - \chi_t}{2m^2} + m^2 I^{(1)}[f^2] = d_1 m^2 + d_2 m^4 + o(m^4)$$

Follow already from general m^2-analyticity argument for χ_t and $\mathcal{C}_{u\bar{u}} = \frac{\partial \mathcal{C}_{\bar{u}}}{\partial m^2}$

$$\lim_{m\to 0}\lim_{\mathrm{V}_4\to\infty} \frac{8}{m^2}\int_0^2 d\lambda\, \hat{f}(\lambda; m)\left\langle Q^2 \rho_A(\lambda)\right\rangle_c = \lim_{m\to 0}\left[\mathcal{C}_{uu} - \mathcal{C}_{ww}\right]$$

$\mathrm{U}(1)_A$ order parameter $\mathcal{C}_{uu} - \mathcal{C}_{ww} \propto \langle P_a^2 P_b^2\rangle - \langle S_a^2 S_b^2\rangle$ $(a \neq b)$, possible $\mathrm{U}(1)_A$ breaking from correlation between zero and nonzero modes

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Second-order Constraints - III

One more constraint on topology

$$\frac{b_{Q^4} - \chi_t}{m^2} = O(m^2)$$

First nontrivial cumulant

$$b_{Q^4} \equiv \lim_{\mathrm{V}_4 \to \infty} \frac{\langle Q^4 \rangle - 3\langle Q^2 \rangle^2}{\mathrm{V}_4}$$

$b_{Q^4} = \chi_t + O(m^4)$, same as gas of non-interacting (anti)instantons

$$Q = n_i - n_a \qquad P_{n_i}(n) = P_{n_a}(n) = e^{-\frac{\chi_t}{2}} \frac{1}{n!} \left(\frac{\chi_t}{2}\right)^n$$

True for all cumulants, see [Kanazawa, Yamamoto (2014)]

$\Rightarrow$ expect early onset of dilute-gas behaviour for physical masses, as in pure gauge [Bonati *et al.* (2013)]

Summary and outlook

Chiral symmetry restoration in the $N_f \geq 2$ massless limit **requires**

$$\text{scalar/pseudoscalar susceptibilities} = \sum_{n=0}^{\infty} m^{2n} a_n$$

For $N_f = 2$:

- $\mathrm{SU}(2)_A$ restoration compatible with $\mathrm{U}(1)_A$ breaking, but requires singular behaviour at zero of eigenvalues one and two-point functions
- m^2-analyticity requires singular peak to tend to m^4/λ as $m \to 0$
- $\mathrm{U}(1)_A$ breaking requires near-zero modes are **not** localised
  – a second near-zero mobility edge?
- $\mathrm{SU}(2)_A$ restoration requires same $P(Q)$ as ideal instanton gas

Open issues:

- higher-order constraints?
- test against numerical results

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GW Determinant in the Presence of Sources

Spectrum of $D_{\rm GW}$ with $R = \frac{1}{2}$: circle in the complex plane ▸ back

Real $\mu_n = 0, 2$, complex pairs $\mu_n, \mu_n^* = \frac{\lambda_n^2}{2} \pm i\lambda_n \sqrt{1 - \frac{\lambda_n^2}{4}}$, $\lambda_n \in (0, 2)$

$$e^{F(J;m)} = \frac{\det M(J;m)}{\det M(0;m)} = e^{N_0 L(u,w,\tilde{u};m)} e^{iQA(u,w,\tilde{u};m)} e^{X(u,w,\tilde{u};m)}$$

$$\det M(0,m) = m^{2N_0} 4^{N_2} \prod_{n=1}^{N} \left(\lambda_n^2 + h(\lambda_n) m^2\right)^2$$

$$L(u,w,\tilde{u};m) = \frac{1}{2} \ln \left(\left(1 + \frac{u-w}{m^2}\right)^2 + \frac{\tilde{u}^2}{m^4} \right) \qquad A(u,w,\tilde{u};m) = \arctan \frac{\tilde{u}}{m^2 + u - w}$$

$$e^{X(u,w,\tilde{u};m)} = \prod_{n=1}^{N} \left[1 + 2 \left(f(\lambda_n; m) u + \tilde{f}(\lambda_n; m) w \right) + f(\lambda_n; m)^2 \left((u-w)^2 + \tilde{u}^2 \right) \right]$$

$$f(\lambda) = \frac{h(\lambda)}{\lambda^2 + m^2 h(\lambda)} \qquad \tilde{f}(\lambda) = f(\lambda) - 2m^2 f(\lambda)^2$$

$$\ln \langle e^F \rangle = \langle F \rangle + \frac{1}{2!} \left(\langle F^2 \rangle - \langle F \rangle^2 \right) + \frac{1}{3!} \left(\langle F^3 \rangle - 3\langle F^2 \rangle \langle F \rangle^2 + 2\langle F \rangle^3 \right) + \ldots$$