Geometric Models of Matter

Guido Franchetti

School of Mathematical & Computer Sciences, Heriot-Watt University

Wednesday 1st October 2014

The idea

- Atiyah, Manton and Schroers, “Geometric models of matter”, Proc. R. Soc. Lond. A 468 (2012): model particles with 4-manifolds.
- Static models ⇔ Riemannian 4-manifolds.
  Time and dynamics still to be included in the picture.
- Conserved quantum numbers ⇔ Topology E.g. electric charge encoded in the asymptotic topology.
  Dynamics ⇔ Geometry E.g. energy related to curvature.
- Unconventional approach to Particle Physics, playground of interesting physical and mathematical ideas. Could eventually clarify the uneasy relation between Quantum Mechanics and Geometry.

Some motivation

- Classical field theory provides interesting insight/approximations of some quantum properties. Particularly relevant are topological solitons: instantons, monopoles, ...
- Conserved quantum numbers (e.g. baryon number) related to topological properties (typically degree) of fields having domain/target which are topologically non-trivial.
  E.g. monopoles, Skyrmions.
- Idea: substitute maps between non-trivial manifolds by the manifolds themselves.

Why 4 dimensions?

4D Physics have often proved useful (or essential) to clarify the apparently 3D Physics of our world.

- Minkowski spacetime and electromagnetism.
- Kaluza-Klein construction.
- Approximate Skyrmion solutions by taking the holonomy of 4D instantons.

The topology of 4-dimensional manifolds is known to be extremely rich - much more than in any other dimension, which could help accommodating the large variety of known particles. We cannot promise we will never go up in dimensions though!

Single Particle Models

Asymptotic Structure

Physical empty 3-space is represented by flat $\mathbb{R}^3$.

Non-compact	Asymptotic circle fibration over $\mathbb{R}^3$.
Compact	Embedded 3-surface X where the 4-manifold intersects physical 3-space.

Rotational invariance

$SO(3)$ acts isometrically on the 4-manifold.

Non-compact	The group action asymptotically is a bundle map covering the usual $SO(3)$ action on $\mathbb{R}^3$.
Compact	The group action fixes the 3-surface X.

Single-particle Models — Examples

	Charge	Particle	Notes
Taub-NUT	-1	Electron	No core structure
Atiyah-Hitchin	$+1$	Proton	Core structure: bolt
$\mathbb{C}P^2$	0	Neutron	Fubini-Study metric

Some details on Taub-NUT

Non compact: topologically $\mathbb{R}^4$.
Asymptotic structure: S^3, circle bundle over S^2 (Hopf fibration).

$$\mathrm{d}s^2 = \left(1 + \frac{1}{2r}\right)\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) + \left(1 + \frac{1}{2r}\right)^{-1}\left(\mathrm{d}\psi + \frac{1}{2}\cos\theta\,\mathrm{d}\phi\right)^2$$

$SU(2)$ isometry generated by the usual Killing vector fields.
Additional $U(1)$ isometry generated by $\partial/\partial\psi$. The fixed point, $r = 0$, of this isometry is interpreted as the position of the electron.

Electrical Charge

- Neutral particles: trivial fibration (or compact M).
- Electrically charged particles: non-trivial asymptotic fibration.

A non-compact manifold is required to asymptotically approach a circle fibration over $\mathbb{R}^3$. If the asymptotic fibration is oriented and c_1 is its first Chern number, then

$$\boxed{Q = -c_1.}$$

c_1 can be calculated by integrating the field strength F of any connection defined on the asymptotic $U(1)$-bundle over the base B of the fibration,

$$c_1 = -\frac{1}{2\pi} \int_B F.$$

We Need Multi Particle Models

Eventually we would like to be able to study interactions.

- Need more than one particle! Introduce multi particle models. A natural candidate, multi Taub-NUT, is a particular example of Gravitational Instantons. Therefore, consider the suitability of Gravitational Instantons as models for particle systems.
- Need a notion of energy. We will construct energy functionals in terms of geometrical quantities defined on our multi particle models.

Gravitational Instantons

- Hyperkähler 4-manifolds $\Rightarrow$ Ricci-flat, self-dual.
- Can be either compact or non-compact. If non-compact, the Riemann tensor is required to approach zero at infinity. The only compact ones are T^4 (the 4-torus) and K3.
- Non-compact examples can be constructed using the Gibbons-Hawking ansatz, G. Gibbons and S. W. Hawking, "Gravitational Multi-Instantons", Phys. Lett. B 78 (1978).
- The Gibbons-Hawking metric is

$$\mathrm{d}s^2 = V\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) + V^{-1}\left(\mathrm{d}\psi + \alpha\right)^2,$$

V is harmonic and independent of ψ and (locally) $\mathrm{d}\alpha = *_3\mathrm{d}V$.

Classification of Non-compact Gravitational Instantons

Class	Volume growth	Example
ALE	r^4	Eguchi-Hanson
ALF	r^3	Taub-NUT
ALG	$r^a, 2 \leq a < 3$	
ALH	$r^a, a < 2$	

ALF

- A_k. Discovered by S. W. Hawking, "Gravitational Instantons", Phys. Lett. A 60 (1977). Metric explicitly known.
- D_k. Discovered by A. S. Dancer, "Dihedral Singularities and Gravitational Instantons", J. Geom. Phys. 12 (1993). Metric known only implicitly.

ALF A_k

Infinite family labelled by the integer $k \geq 0$.

TOPOLOGY	$k = 0$ $\mathbb{C}^2$ $k > 0$ minimal resolution of $\mathbb{C}^2/\mathbb{Z}_{k+1}$
METRIC	Gibbons-Hawking form $\mathrm{d}s^2 = V\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right) + V^{-1}\left(\mathrm{d}\psi + \alpha\right)^2$ $V = 1 + \frac{1}{2}\sum_{i=1}^{k+1}\left(\|\|p - p_i\|\|\right)^{-1}$
NUTs	$\{p_i\}$ $k+1$ distinct points in $\mathbb{R}^3$. Fixed points of the $U(1)$ isometry generated by $\partial/\partial\psi$.

A_0, also known as Taub-NUT, is the model of the electron.

Asymptotic topology

$k=0$: Hopf fibration.

Large r	Base	Fibre
$S^3=\{(z_1,z_2)\in\mathbb{C}^2 \mid |z_1|^2+|z_2|^2=1\}$	S^2	S^1

S^3 points with same ratio z_2/z_1 mapped to the same S^2 point.

$k>0$: $\mathbb{Z}_{k+1}$ action generated by $(z_1,z_2)\mapsto e^{\frac{2\pi i}{k+1}}(z_1,z_2)$.
z_1/z_2 invariant $\Rightarrow$ the asymptotic fibration has the same base, S^2.

A_k large r hypersurfaces: $U(1)$-bundles over S^2.

A_k Charge

- In order to compute c_1, consider the $U(1)$-connection

$$\omega_l = \begin{cases} \mathrm{d}\psi_N + (A_l)_N & \text{on } U_N \\ \mathrm{d}\psi_S + (A_l)_S & \text{on } U_S \end{cases}, \quad \left.\begin{matrix} (A_l)_N \\ (A_l)_S \end{matrix}\right\} = \frac{l}{2}\left(\cos\theta \mp 1\right)\mathrm{d}\phi.$$

  A $U(1)$-bundle over S^2 with connection form ω_l has $c_1 = l$.
- Asymptotic metric: $\mathrm{d}s^2 = V(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2) + 4V^{-1}\left(\omega_{k+1}\right)^2$

$$Q_{A_k} = -(k+1).$$

- The value of the charge and the fact the A_k looks like the model for an electron (A_0) close to each NUT suggest to consider it as a model for a system of $k+1$ electrons.

ALF D_k

Infinite family labelled by the integer $k \geq 0$.

Topology	$k = 0$ retracts onto $\mathbb{R}P^2$
	$k = 1$ retracts onto S^2
	$k = 2$ minimal resolution of $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$
	$k \geq 3$ minimal resolution of $\mathbb{C}^2/D^*_{k-2}$
As. Metric	$ds^2 = V\left(dr^2 + r^2 d\Omega^2\right) + V^{-1}\left(d\psi + \alpha\right)^2$
	$V = 1 - \frac{2}{\lVert p \rVert} + \frac{1}{2}\sum_{i=1}^{k}\left(\frac{1}{\lVert p - p_i \rVert} + \frac{1}{\lVert p + p_i \rVert}\right)$
	$\mathbb{Z}_2$ identification $(\theta, \phi, \psi) \sim (\pi - \theta, \phi + \pi, -\psi)$

It is possible to move a pair of NUTs to the origin without making the manifold singular or altering its topology. Doing so for $k = 1$ one obtains the Atiyah-Hitchin manifold, the model of the proton.

Moduli Space Interpretation

	V	$\psi \in$
D_0	$1 - \frac{2}{\|\|p\|\|}$	$[0, 2\pi)$
AH	$1 - \frac{1}{\|\|p\|\|}$	$[0, 2\pi)$
	$1 - \frac{2}{\|\|p\|\|}$	$[0, 4\pi)$
D_1	$1 - \frac{2}{\|\|p\|\|} + \frac{1}{\|\|p-p_1\|\|} + \frac{1}{\|\|p+p_1\|\|}$	$[0, 2\pi)$

- D_0: true moduli space of centred charge 2 $SU(2)$ monopoles.
- Atiyah-Hitchin: simply connected double cover of D_0.
- D_k: moduli space of $(2, k)$ centred $SU(3)$ monopoles in the infinite mass limit of the $(0, 1)$ monopoles. See S. A. Cherkis and A. Kapustin "Singular monopoles and supersymmetric gauge theories in three dimensions", Nucl. Phys. B 525 (1998).

Asymptotic topology

$k \geq 3$: D^*_{k-2} action generated by

$$(z_1, z_2) \mapsto \mathrm{e}^{\frac{i\,\pi}{k-2}}(z_1, z_2),$$
$$(z_1, z_2) \mapsto i(\bar{z}_2, -\bar{z}_1)$$

The second generator identifies a point on S^2 with its antipodal point. $\Rightarrow$ Large r hypersurfaces are circle bundles over $\mathbb{R}P^2$.

$k = 2$: $\frac{\mathbb{R}^3 \times S^1}{\mathbb{Z}_2}$ with $\mathbb{Z}_2$ action generated by $(\mathbf{x}, \psi) \mapsto (-\mathbf{x}, 2\pi - \psi)$.

$k = 0, 1$: same as. topology as $k = 4, 3$ but opposite orientation.

D_k large r hypersurfaces: unoriented circle bundles over $\mathbb{R}P^2$.

D_k Charge

- The leading asymptotic form of the metric can be written $\mathrm{d}s^2 = V(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2) + V^{-1}\left(\omega_{2(k-2)}\right)^2$. However the asymptotic circle bundle is not oriented and it is not possible to define its first Chern number.
- Alternative definition (equivalent for oriented fibrations): minus the self-intersection number of the base of the asymptotic fibration. This is equal to the first Chern number of the $U(1)$ asymptotic fibration of the double cover $\overline{D}_k$ of D_k obtained by lifting the $\mathbb{Z}_2$ identification. The form $\omega_{2(k-2)}$ is a connection form on $\overline{D}_k$, therefore

$$\boxed{Q_{D_k} = -\frac{1}{2} \cdot 2(k-2) = 2 - k.}$$

D_k Particle Interpretation

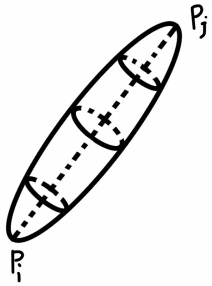
$$V = \underbrace{1 - \frac{2}{||p||}}_{\substack{\text{asymptotic } D_0 \\ \text{charge } +2}} + \overbrace{\underbrace{\frac{1}{2}\sum_{i=1}^{k}\left(\frac{1}{||p-p_i||} + \frac{1}{||p+p_i||}\right)}_{\substack{k \text{ pairs of mirror symmetric } A_0 \text{ NUTs} \\ \text{each pair has charge } -1}}}^{k \text{ ELECTRONS}}$$

$$V\big|_{p_k=0} = \overbrace{\underbrace{1 - \frac{1}{||p||}}_{\substack{\text{asymptotic AH} \\ \text{charge } +1}}}^{\text{PROTON}} + \overbrace{\underbrace{\frac{1}{2}\sum_{i=1}^{k-1}\left(\frac{1}{||p-p_i||} + \frac{1}{||p+p_i||}\right)}_{\substack{k-1 \text{ pairs of } A_0 \text{ NUTs} \\ \text{each pair has charge } -1}}}^{k-1 \text{ ELECTRONS}}$$

Either a particle of charge $+2$ and $k\ e^-$, or a proton and $k-1\ e^-$.

Topologically Preferred 2-cycles

$$H_p(M_k, \mathbb{Z}) = \begin{cases} \mathbb{Z}^k & \text{if } p = 2 \\ \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$ where M_k is either A_k or D_k.



Build generators. We follow A. Sen, "A note on enhanced gauge symmetries in M- and string theory", JHEP 09 (1997).

- Take the (Euclidean) line connecting two NUTs and erect a circle of radius $1/V(r)$ at each point of the line.
- NUTs are fixed points of the circle action generated by $\partial/\partial\psi$ so the surface is topologically a 2-sphere.

Minimal Area

- The induced metric is $\mathrm{d}\sigma^2 = V\,\mathrm{d}r^2 + V^{-1}\,\mathrm{d}\psi^2$.
- The area A of the 2-cycle $S_{i,\,j}$ is

$$A\left(S_{i,\,j}\right) = \int_0^{2\pi} V^{-1}\mathrm{d}\psi \int_{p_i}^{p_j} V\mathrm{d}r = 2\pi\,||p_i - p_j||,$$

  proportional to the distance between the two NUTs.
- This is exact for A_k, approximate for D_k since the metric is only valid asymptotically.
- If we had taken any other curve connecting p_i and p_j, the area would have been 2π times the Euclidean length of the curve, therefore the 2-cycles constructed have minimal area.
- Get Coulomb energy by summing the inverse areas of appropriate 2-cycles.

First Energy Functional

- It is possible to choose basis 2-cycles which intersect according to the Cartan matrix of the corresponding Lie algebra.

	Simple roots	All roots
A_{k-1}	$\{e_1 - e_2, \ldots, e_{k-1} - e_k\}$	$\{\pm(e_i - e_j)\}$, $i \neq j$
D_k	$\{e_1 - e_2, \ldots, e_{k-1} - e_k, e_{k-1} + e_k\}$	$\{\pm(e_i \pm e_j)\}$, $i \neq j$

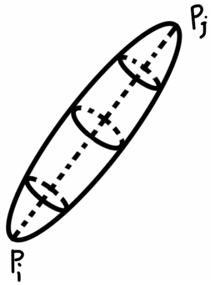
- Associate the 2-cycle $S_{i,\pm j}$, connecting the NUTs $\pm p_i$ to the NUTs $\pm p_j$, to the root $e_i \pm e_j$.
- Simple roots are not invariant under the Weyl group, all pairs of NUTs should play an equal rôle. Therefore, sum π times the inverse area of the 2-cycles corresponding to all the roots.

$$E_{A_{k-1}} = \sum_{i<j}^{k} \frac{1}{||p_i - p_j||}$$

$$E_{D_k} = \sum_{i<j}^{k} \left(\frac{1}{||p_i - p_j||} + \frac{1}{||p_i + p_j||} \right)$$

- $E_{A_{k-1}}$ is the Coulomb interaction energy of k electrons.
- E_{D_k} only gives electron-electron interactions.
- The construction only involves pairs of NUT, $k \geq 2$, so it does not run into self-energy problems.

Gaussian Curvature at NUTs



- NUTs are geometrically preferred points.
- Calculate the Gaussian curvature of the 2-cycle $S_{i,\,j}$, $i \neq j$ at the points p_i and p_j.
- The metric is $\mathrm{d}\sigma^2 = V\,\mathrm{d}x^2 + V^{-1}\,\mathrm{d}\psi^2$ (surface of revolution).
- The Gaussian curvature K is independent of ψ and is given by

$$K = -\frac{1}{2}\frac{\partial^2 (V)^{-1}}{\partial x^2}.$$

- Calculating K at an arbitrary point is very cumbersome. Instead, rewrite V^{-1} under a common denominator and take advantage of the fact that many factors are zero at p_i.
- For A_{k-1}, writing $d_i \equiv ||p - p_i||$, $d_{ij} \equiv ||p_i - p_j||$,

$$K = -\frac{1}{2}\frac{1}{(f+g)^3}\Big((f_{xx}g - fg_{xx})(f+g) - 2(f_x + g_x)(f_x g - f g_x)\Big),$$

where $\quad f = \prod_{l=1}^{k} d_l, \quad g = \frac{1}{2}\sum_{l=1}^{k} d_1 \dots \widehat{d_l} \dots d_k.$

- In the limit $p \to p_i$,

$$f \to 0, \quad f_x \to d_{1i} \dots \widehat{d_{ii}} \dots d_{ki}, \quad f_{xx} \to 2\,(d_1 \dots \widehat{d_i} \dots d_k)_x\Big|_{p_i},$$

$$g \to \frac{f_x}{2}, \quad g_x \to \frac{1}{2}(d_1 \dots \widehat{d_i} \dots d_k)_x\Big|_{p_i} + \frac{1}{2}\sum_{j=1,\, j\neq i}^{k} d_{1i} \dots \widehat{d_{ji}} \dots d_{ki}.$$

- For A_{k-1}, the Gaussian curvature of the 2-cycle $S_{i,\,j}$, evaluated at the point p_i, is

$$K_{i,\,j}(p_i) = 4\left(1 + \frac{1}{2}\sum_{l=1, l\neq i}^{k} \frac{1}{||p_l - p_i||}\right).$$

- For D_k, the Gaussian curvature of the 2-cycle $S_{i,\,j}$, evaluated at the point p_i, is

$$K_{i,\,j}(p_i) = 4\left(1 - \frac{2}{||p_i||} + \frac{1}{2}\sum_{l=1,\, l\neq i}^{k}\left(\frac{1}{||p_l - p_i||} + \frac{1}{||p_l + p_i||}\right)\right).$$

Second Energy Functional

In both cases, $K_{i,j}(p_i) = K_{i,l}(p_i)\ \forall l \neq i$. We can associate the Gaussian curvature $K(p_i)$ to the point p_i without reference to the 2-cycle used to calculate it. Defining the energy functional to be the sum of $K(p_i)/4$ over the NUTs $p_1, \ldots p_k$ we have, for $k \geq 2$

$$E_{A_{k-1}} = k + \sum_{i<j=1}^{k} \frac{1}{||p_i - p_j||},$$

$$E_{D_k} = k - \sum_{i=1}^{k} \frac{2}{||p_i||} + \sum_{i<j=1}^{k} \left(\frac{1}{||p_i - p_j||} + \frac{1}{||p_i + p_j||} \right).$$

Comments

- The Gaussian energy functional for A_{k-1} differs from the inverse area one only by the addition of a constant term equal to the number of electrons.
- The Gaussian energy functional for D_k has both an electron-electron part, equal to the inverse of area functional, and a part corresponding to the Coulomb interactions between the positively charged particle and the electrons.
- If all the NUTs are far from the origin, we can neglect the terms $1/\left\|p_i + p_j\right\|$ and E_{D_k} reduces to the Coulomb interaction energy of a particle of charge $+2$ and k electrons plus an additive constant equal to the number of electrons.

D_k Gaussian Energy if $p_k = 0$

- Consider the 2-cycles in the limit $p_k = 0$. If all the other NUTs are far from the origin, the metric still describes accurately the geometry of the 2-cycles around $\{p_i\}$, $i \neq k$.
- Calculating the Gaussian energy functional as before, and dropping the contribute of the NUT at the origin we get

$$E_{D_k} = k-1-\sum_{i=1}^{k-1} \frac{1}{||p_i||}+\sum_{i<j=1}^{k-1} \left(\frac{1}{||p_i - p_j||} + \frac{1}{||p_i + p_j||} \right).$$

- We expect corrections due to the contribution of the origin to be related to the rest mass of the positively charged particle.

Physical Units

So far we have used geometrical units such that the length of the circles in the asymptotic fibration is 2π. In physical units, chosen so that this length is $2\pi r_e$, the Gaussian energy functionals become

$$E_{A_{k-1}} = k\, m_e c^2 + \sum_{i<j=1}^{k} \frac{e^2}{||p_i - p_j||},$$

$$E_{D_k} = k\, m_e c^2 - \sum_{i=1}^{k} \frac{2e^2}{||p_i||} + \sum_{i<j=1}^{k} \left(\frac{e^2}{||p_i - p_j||} + \frac{e^2}{||p_i + p_j||} \right),$$

$$E_{D_k}\big|_{p_k=0} = (k-1)m_e c^2 - \sum_{i=1}^{k-1} \frac{e^2}{||p_i||} + \sum_{i<j=1}^{k-1} \left(\frac{e^2}{||p_i - p_j||} + \frac{e^2}{||p_i + p_j||} \right).$$

What Have We Done?

- Introduced two infinite families, ALF A_k and D_k Gravitational Instantons, as candidates for particle systems and computed their charge. A_k models $k + 1$ electrons, D_k either a particle of charge $+2$ and k electrons, or a proton and $k - 1$ electrons.
- Constructed energy functionals which reproduce the Coulomb interaction energy of the appropriate particle system and, in one case, also the rest mass of the electrons. The structures considered naturally involve couples of particles, so do not cause any self-energy problem.
- Considered the properties of 2-dimensional substructures. The results obtained suggest that these structures may play an important rôle in this geometrical approach.

For the Future

- Investigate the exact D_k metric and find what are the energy corrections if the NUTs are not far from the origin.
- Understand the relation between the two different particle interpretations of D_k – this also requires a better understanding of the exact metric around the origin.
- Multi-baryon systems might need to be modelled by non self-dual manifolds. This brings in other interesting candidates, e.g. Euclidean Schwarzschild, Taub-Bolt.
- Might want to use a non-compact manifold as a model for the neutron. Euclidean Schwarzschild is a candidate but it admits orientation reversing isometries. Also, the $SU(2)$ orbit structure is very different from that of AH.

Thank you very much for your attention

Bibliography

- G. Franchetti and N. Manton. A_k and D_k gravitational instantons as models for charged multi-particle systems.
  J. High Energy Phys., 03:072, 2013, arXiv:1301.1624
- M. Atiyah, N. S. Manton, and B. J. Schroers. Geometric models of matter.
  Proc. R. Soc. Lond. A, 468:1252–1279, 2012
- G. Gibbons and S. W. Hawking. Gravitational multi-instantons.
  Phys. Lett. B, 78:430–432, 1978
- S. W. Hawking. Gravitational instantons.
  Phys. Lett. A, 60:81–83, 1977
- A. S. Dancer. Dihedral singularities and gravitational instantons.
  J. Geom. Phys., 12:77–91, 1993
- S. A. Cherkis and A. Kapustin. Singular monopoles and supersymmetric gauge theories in three dimensions.
  Nucl. Phys. B, 525:215–234, 1998
- A. Sen. A note on enhanced gauge symmetries in M- and string theory.
  J. High Energy Phys., 09:001, 1997