Beyond-all-order effects in a fifth order Korteweg–de Vries equation

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in collaboration with

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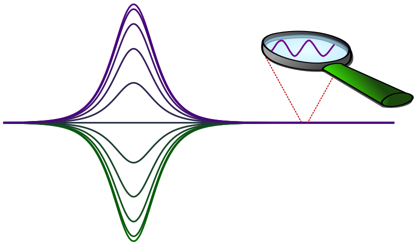
Motivation: massive nonlinear scalar fields



Oscillating core and extremely small amplitude radiating tail

- flat background: oscillon
- Einstein gravity: oscillaton

Adding incoming radiation $\longrightarrow$ time periodic standing-wave tail

Small parameter: core amplitude ϵ

$\longrightarrow$ tail amplitude is exponentially small $\quad \alpha \approx \frac{a}{\epsilon^k} \exp\left(-\frac{b}{\epsilon}\right)$

faster than any power law – “beyond-all-orders” effect

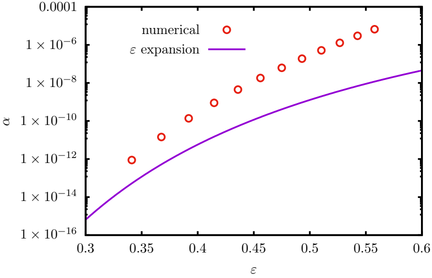
Segur-Kruskal method (1987)

- extension to the complex plane
- solve the "inner equations" close to the nearest pole
  - numerically
  - Borel summation (Pomeau, Ramani and Grammaticos 1988)
- technically rather complicated calculation

$\alpha \approx \frac{a}{\epsilon^k} \exp\left(-\frac{b}{\epsilon}\right)$ is only a leading order result for tail amplitude

$\longrightarrow$ analytic calculation is only valid for very small ϵ amplitudes

Numerical simulations can be done only for relatively large ϵ when the tail amplitude α is not extremely small



– this plot is for gravitational oscillatons

  – at least factor 20 difference

– similar (less drastic) results for flat-space oscillons

Expected corrections: $\alpha \approx \frac{a}{\epsilon^k}(1 + c_1\epsilon + c_2\epsilon^2 + \ldots)\exp\left(-\frac{b}{\epsilon}\right)$

no results for c_n are known for the scalar field problem

– time-periodic solutions, coupled differential equations for Fourier components: $\phi = \phi_0 + \phi_1\cos(\omega t) + \phi_2\cos(2\omega t) + \phi_3\cos(3\omega t)\ldots$

Study a time independent system first, where there is also core and exponentially small tail

Generalized KdV equation

Stationary Korteweg–de Vries (KdV) equation
with a 4-th derivative term

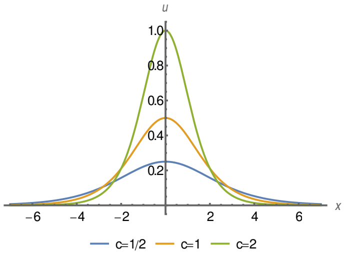
$$\boxed{\epsilon^2 u_{xxxx} + u_{xx} + 3u^2 - cu = 0}$$

c and ϵ parameters, ϵ is not necessarily very small
– ordinary differential equation for $u \equiv u(x)$

Can be obtained from the fifth-order KdV equation

$$\epsilon^2 u_{yyyyy} + u_{yyy} + 6uu_y + u_t = 0$$

looking for stationary solutions moving with speed c to the right,
$x = y - ct$, and integrating once



For $\epsilon = 0$ and $c > 0$ the stationary KdV equation

$$u_{xx} + 3u^2 - cu = 0$$

has the solitary wave (or soliton) solution

$$u = \frac{c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}x\right)$$

the amplitude is always positive

$\longrightarrow$ only elevation waves

KdV equation can be obtained when studying surface water waves for shallow depth and long wavelength

Long wavelength sinusoidal waves move right with speed $c_s = \sqrt{hg}$
– where h is the average water depth, g is the gravity of Earth

Solitary waves move with higher speed c_{ph}, such that $c \sim c_{ph} - c_s$
– they are supercritical

KdV equation can also describe waves with surface tension σ

$\left(\frac{1}{3} - B\right) u_{xx} + 3u^2 - cu = 0$

where $B = \dfrac{\sigma}{\rho g h^2}$ is the Bond number

(ρ fluid density, h average depth, g gravity of Earth)

– signature of u_{xx} term changes when $B > \frac{1}{3}$

– for $B > \frac{1}{3}$ there are depression solitary waves with subcritical speed and no oscillating tail

– for $B \approx \frac{1}{3}$ a fourth derivative should be added to the equation

We are interested in tails, so we assume B is close to but below $\frac{1}{3}$

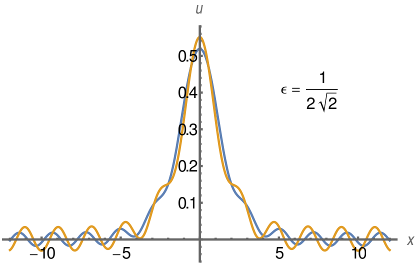
$$\epsilon^2 u_{xxxx} + u_{xx} + 3u^2 - cu = 0$$

How solitons change for small $\epsilon > 0$?

Equation $\epsilon^2 u_{xxxx} + u_{xx} + 3u^2 - cu = 0$ remains invariant for the rescalings

$$u = s^2 \tilde{u} \ , \quad x = \frac{1}{s}\tilde{x} \ , \quad c = s^2 \tilde{c} \ , \quad \epsilon = \frac{1}{s}\tilde{\epsilon}$$

$\longrightarrow$ we only need to perform numerical simulations for $c = 1$



Solutions with tail on both sides are necessarily symmetric

Form of the tail:

$$u = \alpha \sin\left(\frac{kx}{\epsilon} - \delta\right)$$

α is never zero for $\epsilon > 0$

linearized equation:

$$k^4 - k^2 - c\epsilon^2 = 0$$

$$k = \sqrt{\frac{1}{2}\left(1 + \sqrt{1 + 4c\epsilon^2}\right)} = 1 + \frac{c}{2}\epsilon^2 + \dots$$

for small ϵ wavelength decreases proportionally to ϵ

For each ϵ there is a minimal amplitude α_m belonging to phase δ_m
– α_m determines energy loss and lifetime
– amplitude for any phase δ can be obtained as $\alpha = \dfrac{\alpha_m}{\cos(\delta - \delta_m)}$

Numerical minimization to get α_m
– configurations with several different δ must be calculated

There is an asymptotic expansion

$$\delta_m = c_1\epsilon + c_5\epsilon^5 + c_7\epsilon^7 + c_9\epsilon^9 + \dots$$

c_n can be calculated to high orders
– minimization can be avoided in numerical calculations for small ϵ

ϵ	α_m	δ_m
2^{-1}	$4.8 \cdot 10^{-2}$	1.2
2^{-2}	$1.53 \cdot 10^{-3}$	0.749
2^{-3}	$3.25 \cdot 10^{-8}$	0.373
2^{-4}	$1.94 \cdot 10^{-18}$	0.187
2^{-5}	$1.27 \cdot 10^{-39}$	0.0937
2^{-6}	$1.17 \cdot 10^{-82}$	0.0469
2^{-7}	$2.30 \cdot 10^{-169}$	0.0234
2^{-8}	$2.13 \cdot 10^{-343}$	0.0117

Numerical method

We look for solutions reflection symmetric at $x = 0$

Match to linearized tail $u = \alpha \sin\left(\frac{kx}{\epsilon} - \delta\right)$ at outer boundary
– two boundary conditions at $x = L$

Choose some δ, the amplitude α will come out as a result

Rescale by $x = L\,\tilde{x}$ to make computational interval $0 \leq \tilde{x} \leq 1$
– expand in even indexed Chebyshev polynomials $T_{2n}(\tilde{x})$

Chebyshev polynomial expansion is merely a Fourier cosine series in disguise (J.P. Boyd's book: Chebyshev and Fourier Spectral Methods)

– define θ by $\tilde{x} = \cos\theta$ then $T_n(\tilde{x}) = \cos(n\theta)$

Two equivalent ways to represent function u by N numbers:

- Fourier (i.e. Chebyshev) components U_n
- values u_n at collocation points $\theta_n = \dfrac{\pi n}{2(N-1)}$

Transition between the two sets is by matrix multiplication
(or Fast Fourier Transform) without precision loss
– differentiation is by matrix multiplication on Fourier components
– multiplication can be calculated using collocation values

Linear equations can be solved in one step (matrix inversion)

Iterative method is applied for nonlinear equations
– solving equation linearized around current approximation
– usually ~ 15 steps is enough

Error decreases exponentially when increasing the number of collocation points N
– one-dimensional problem: results may converge to ~ 100 digits

Central amplitude ~ 1, tail amplitude $\alpha \sim 10^{-a}$
– to get α for b digits we need $a + b$ digit numbers

Arb – C library for arbitrary-precision floating-point ball arithmetic

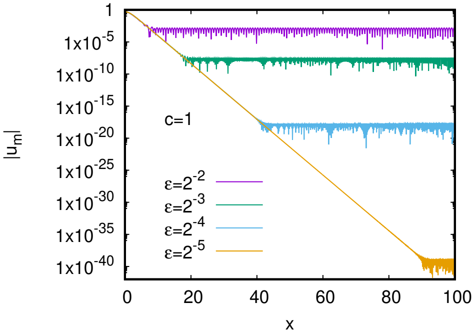
- ball: error of each long number is represented by a machine precision number (no significant decrease in speed)
- $c = a + b \quad \longrightarrow \quad \mathrm{arb\_add(c, a, b)}$
- matrix multiplication or inversion is just a single command
- used by Mathematica, Maple, SageMath...

CLN – Class Library for Numbers

- C++ library, easier to write codes – significantly slower
- no ready matrix operation routines

largest resolution used: $N = 5000$ collocation points, 100 digits
– running time ~ 12 hours

Minimal tail solutions



Downwards spikes:
zero crossings in tail

Shape and decay rate of core
seem to be ϵ independent

Linearized equation:
$\epsilon^2 u_{xxxx} + u_{xx} + \cancel{3u^2} - cu = 0$
substitute $u = \exp(-2\gamma x)$
$16\gamma^4\epsilon^2 + 4\gamma^2 - c = 0$

$$\gamma = \sqrt{\frac{1}{8\epsilon^2}\left(-1 + \sqrt{1 + 4c\epsilon^2}\right)} = \frac{\sqrt{c}}{2}\left(1 - \frac{c}{2}\epsilon^2 + \dots\right)$$

Alternative viewpoint: assume that γ is independent of ϵ ,
$\longrightarrow$ c depends on ϵ according to $c = 4\gamma^2 + 16\gamma^4\epsilon^2$

Expansion for the core region

We intend to solve

$$\epsilon^2 u_{xxxx} + u_{xx} + 3u^2 - cu = 0 \quad , \qquad c = 4\gamma^2 + 16\gamma^4\epsilon^2$$

for small ϵ, where γ is a constant (independent of ϵ)

– two parameters: $(\epsilon, c) \longrightarrow (\epsilon, \gamma)$ – we get all solutions

– decay rate of the core is ϵ independent

– position of the poles on the complex plane is ϵ independent

We look for solution as a formal expansion: $u = \sum_{n=0}^{\infty} u_n \epsilon^{2n}$

This will not be able to describe the tail $u = \alpha \sin\left(\frac{kx}{\epsilon} - \delta\right)$ where $k = \sqrt{1 + 4\gamma^2\epsilon^2}$, since α is exponentially small in ϵ

To leading order: $u_0 = 2\gamma^2 \mathrm{sech}^2(\gamma x)$ is the KdV soliton

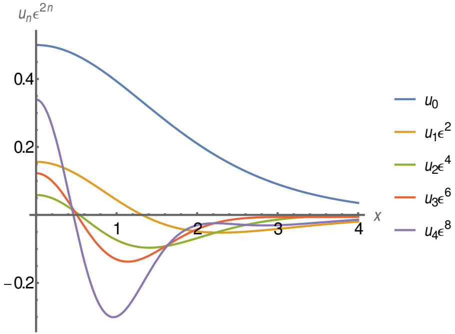
– all u_n are $n+1$ degree polynomials in $\mathrm{sech}^2(\gamma x) = \dfrac{1}{\cosh^2(\gamma x)}$

$$u = \sum_{n=0}^{\infty} u_n \epsilon^{2n} \quad \text{where} \quad u_n = \gamma^{2n+2} \sum_{j=1}^{n+1} u_{n,j} \,\mathrm{sech}^{2j}(\gamma x)$$

$u_{n,j}$

$n \downarrow \;\; j \rightarrow$	1	2	3	4
0	2	-	-	-
1	-20	30	-	-
2	60	-930	930	-
3	-2472	21036	-66216	49662

$u_{n,j}$ can be calculated fast up to $n \approx 100$ by a Mathematica code



contribution of terms in

$$u_N = \sum_{n=0}^{N} u_n \epsilon^{2n}$$

for $\epsilon = \gamma = \frac{1}{2}$

optimal order of truncation
in this case is $N_{\rm opt} = 2$

ϵ expansion is not convergent
it is an asymptotic series

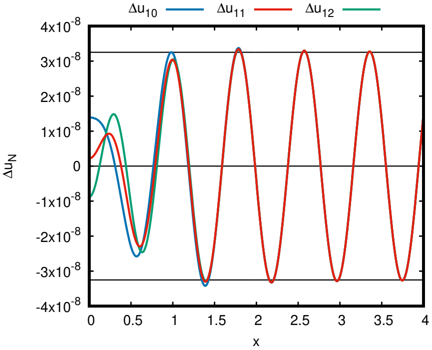
Increasing the number of terms in u_N the approximation gets gradually better until $N = N_{\rm opt}$, and become worse after

– usually the contribution of the $N_{\rm opt}$ term is the smallest (± 1)

– gives a simple rule to decide when to stop summation

ϵ	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$N_{\rm opt}$	2	5	11	24	50	100

$$N_{\rm opt} \sim 1/\epsilon$$



$\Delta u_N = u_m - u_N$ where

u_m is the minimal tail numerical solution ,

$$u_N = \sum_{n=0}^{N} u_n \epsilon^{2n}$$

$\epsilon = 2^{-3}$, $\alpha_m = 3.25 \cdot 10^{-8}$

optimal truncation:

$N_{\rm opt} = 11$ (red curve)

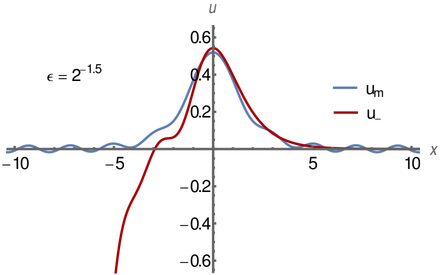
this is well inside the core radius $x \sim 15$

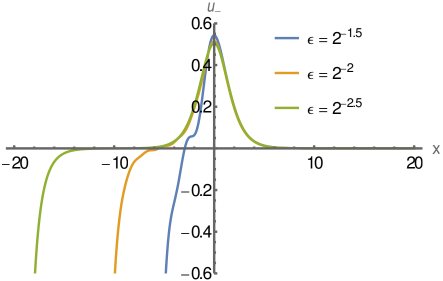
Spectral numerical calculation can be made much more efficient by calculating first the optimal approximation $u_{N_{\rm opt}}$, then solving the nonlinear differential equation for $\Delta u_{N_{\rm opt}} = u - u_{N_{\rm opt}}$

– function remains very small everywhere, but many oscillations

– still need more than 16 digits

Asymmetric solutions





Linear analysis would suggest a solution with no tail on one side and double tail on other side

Exponential decay for $x > 0$ $\longrightarrow$ blow-up at finite $x < 0$

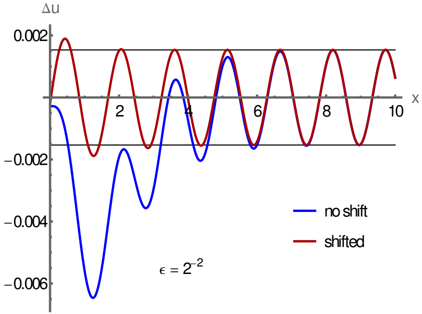
– only the symmetric solution is valid in the full $-\infty < x < \infty$ range

For smaller ϵ the singularity is more distance from the core

Spectral numerical code with compactification can be used

For given ϵ the minimal tail symmetric solution u_m is unique

The right decaying asymmetric solution u_- is unique up to translations



Plot of the difference

$$\Delta u = u_m - u_-$$

If u_- is shifted to the left by 0.022, the difference becomes similar to the sine function

Δu can be calculated very precisely using the WKB method (Wentzel–Kramers–Brillouin)

WKB solution of the linearized problem

Substituting $u \to u + w$ into $\epsilon^2 u_{xxxx} + u_{xx} + 3u^2 - cu = 0$ and linearizing:

$$\epsilon^2 w_{xxxx} + w_{xx} + 6u\, w - cw = 0$$

We only use that the background solution u can be approximated by the core expansion $u = \sum_{n=0}^{\infty} u_n \epsilon^{2n}$ (can be either u_m or u_-)

Look for solution in the form $w = \exp A$, where

$$A = \frac{A_{-1}}{\epsilon} + A_0 + A_1\epsilon + A_2\epsilon^2 + \dots$$

The result:

$$w = \beta \exp\left(\sum_{\substack{n=2 \\ \text{even}}}^{\infty} A_n\epsilon^n\right) \sin\left(\frac{kx}{\epsilon} - \delta_w - \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} A_n\epsilon^n\right)$$

$$w = \beta \exp\left(\sum_{\substack{n=2\\ \text{even}}}^{\infty} A_n \epsilon^n\right) \sin\left(\frac{kx}{\epsilon} - \delta_w - \sum_{\substack{n=1\\ \text{odd}}}^{\infty} A_n \epsilon^n\right)$$

β and δ_w arbitrary constants

$k = \sqrt{1 + 4\gamma^2\epsilon^2}$

$A_1 = 6\gamma \tanh(\gamma x)$, $A_2 = 15\gamma^2 \mathrm{sech}^2(\gamma x)$,

$A_3 = 111\gamma^3 \mathrm{sech}^2(\gamma x) \tanh(\gamma x)$,

$A_4 = \frac{525}{2}\gamma^4 \mathrm{sech}^2(\gamma x) \left[3\mathrm{sech}^2(\gamma x) - 2\right]$, ...

– even indexed A_n give corrections to the amplitude

– odd indexed A_n give corrections to the phase

– asymptotic expansion

– order of optimal truncation is same as for the core expansion

w gives very good approximation to the difference of the symmetric and asymmetric solutions, $\quad w \approx u_m - u_-$

– it is easier to calculate the asymmetric solution u_-
– then w can be used to get approximation for the symmetric solution u_m – especially to the tail

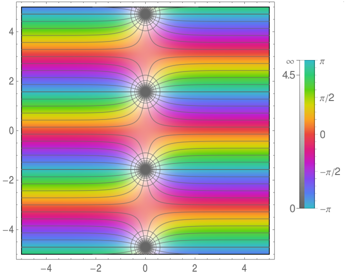
Asymmetry of u_- can be characterized by its third-derivative u_{xxx} at the center $x = 0$ (where $u_x = 0$)

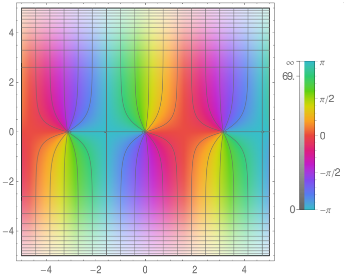
The WKB result can be used to relate the minimal tail amplitude to this third derivative:

$$\alpha_m = y_{xxx}\,\epsilon^2 \left(1 + 5\gamma^2\epsilon^2 + 311\gamma^4\epsilon^4 + 13407\gamma^6\epsilon^6 + \ldots\right)$$

– also an asymptotic expansion

Complex extension (Segur-Kruskal method, 1987)





ComplexPlot of $\operatorname{sech}^2 x$

All terms in the core expansion contain only powers of $\operatorname{sech}^2(\gamma x)$

nearest poles are at $\pm\dfrac{i\pi}{2\gamma}$

ComplexPlot of $\sin x$

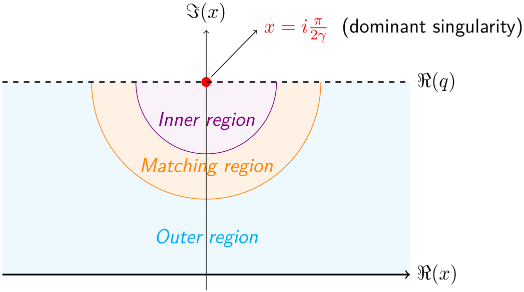
blows up exponentially in the imaginary directions

represents the linear perturbation w

– very small on real axis

– becomes same order as core near the singularity

Tail frequency and blow-up rate grow as $1/\epsilon$



We extend the core expansion

$$u = \sum_{n=0}^{\infty} u_n \epsilon^{2n}$$

and the linearized solution w to the complex x plane

Introduce a rescaled coordinate q to focus on a region close to the first singularity by $x = \dfrac{i\pi}{2\gamma} + \epsilon q$

u is very large there, so we define a rescaled function $v = \epsilon^2 u$

The equation in the inner region becomes

$$v_{qqqq} + v_{qq} + 3v^2 - \epsilon^2 c\, v = 0$$

$v_{qqqq} + v_{qq} + 3v^2 - \epsilon^2 c\, v = 0$

Expand in powers of ϵ, substituting $v = \sum_{n=0}^{\infty} v_n \epsilon^{2n}$

obtain the n-th order inner equations for the functions v_n

Core expansion for real x gives boundary conditions for large $|q|$ (matched asymptotic expansions)

Inner solutions can be determined by

- Borel summation (Pomeau, Ramani and Grammaticos 1988)
- equivalently: Laplace transform (Grimshaw-Joshi 1995)

Complex extension of the WKB solution w can be used to obtain the minimal tail amplitude α_m

The n-th order inner equation should be solved for v_n. Each v_n determines a constant in the tail-amplitude result

$$\alpha_m = \frac{\pi K}{\epsilon^2} \exp\left(-\frac{k\pi}{2\gamma\epsilon}\right)\left(1 - \xi_2\gamma^2\epsilon^2 - \xi_4\gamma^4\epsilon^4 - \xi_6\gamma^6\epsilon^6 - \dots\right)$$

$$k = \sqrt{1 + 4\gamma^2\epsilon^2} \quad , \qquad \gamma \text{ decay rate constant of the core}$$

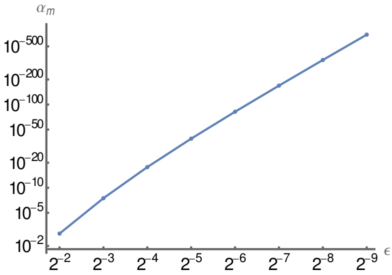
$$v_0 \longrightarrow K = 19.968947 \quad , \quad v_n \longrightarrow \xi_{2n}$$

ξ_2	5
ξ_4	6.5440681
ξ_6	474.41383
ξ_8	4232.4123
ξ_{10}	111053.95
ξ_{12}	1782157.5

Corrections to leading order results only calculated by Grimshaw-Joshi (1995) obtained $\xi_2 = 0$

Spectral numerical result of Boyd (1995): $\xi_2 = 4.985 \pm 0.05$

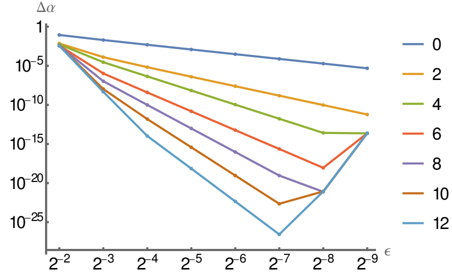
obvious inconsistency, remained unsolved



log-log vs. log plot of minimal tail amplitude α_m as function of ϵ

– at $\epsilon = 2^{-9} = 1/512$

$\alpha_m = 4.5 \cdot 10^{-692}$



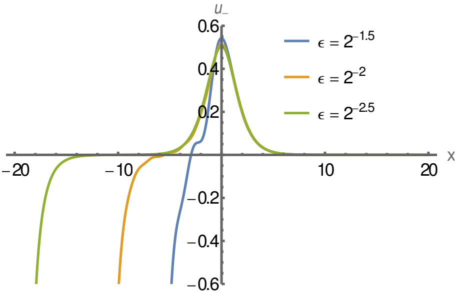
Relative difference of n-th order ϵ-expansion result $\alpha_m^{(n)}$ and numerically obtained α_m

$$\Delta\alpha = \frac{\alpha_m^{(n)} - \alpha_m}{\alpha_m}$$

– power law decrease

– numerical results are less precise for smaller ϵ

Hammersley-Mazzarino method



We look for the asymmetric right decaying solution u_- only for $0 \leq x < \infty$

Rescale variables to make $c = 1$

To agree with Hammersley's notation define $y = 3u$

Our aim is to calculate the third derivative u_{xxx} at the center

Equation to solve: $\epsilon^2 y_{xxxx} + y_{xx} + y^2 - y = 0$

Integrating once: $\epsilon^2 \left(y_x y_{xxx} - \frac{1}{2} y_{xx}^2 \right) + \frac{1}{2} y_x^2 = \frac{1}{2} y^2 - \frac{1}{3} y^3$

Autonomous differential equation, monotonously decreasing y $\longrightarrow$ we could use y as independent variable

$$\epsilon^2\left(y_x y_{xxx} - \frac{1}{2}y_{xx}^2\right) + \frac{1}{2}y_x^2 = \frac{1}{2}y^2 - \frac{1}{3}y^3$$

We use $z = \dfrac{y}{Y}$ as independent variable , where Y is a constant

– define the function $f \equiv f(z)$ by $y_x = -Y\sqrt{f}$

– denote z derivatives as $f_n = \dfrac{\mathrm{d}^n f}{\mathrm{d}z^n}$

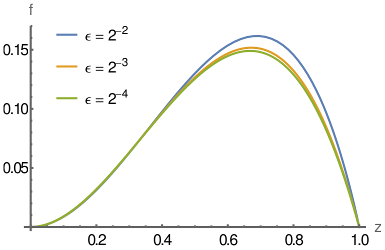
we get a second order differential equation for $f(z)$

$$\epsilon^2\left(f f_2 - \frac{1}{4}f_1^2\right) + f = z^2 - \frac{2}{3}Yz^3$$

for $x \to \infty$ we have $y = 0 \longrightarrow$ infinity corresponds to $z = 0$

at $x = 0$ we have $y \equiv y_c \longrightarrow$ center corresponds to $z = y_c/Y$

– we want to ensure that $Y = y_c$, then the center is at $z = 1$



Look for solution as formal power series

$$f = 4\gamma^2 z^2 \left(1 - \sum_{n=1}^{\infty} c_n z^n\right)$$

γ is the decay constant of the core: $16\gamma^4\epsilon^2 + 4\gamma^2 - 1 = 0$
$\longrightarrow$ appropriate behavior at infinity $z = 0$

The constants c_n can be calculated using a recurrence relation – all determined uniquely by c_1

With the appropriate choice of c_1 the center is at $z = 1$

Hammersley and Mazzarino (1989) showed that the series for f is convergent, and its convergence radius is exactly 1

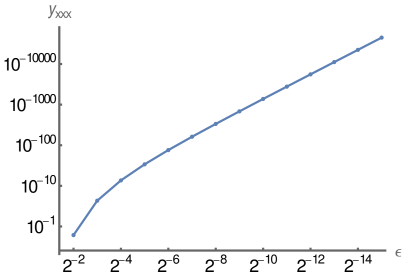
Third derivative y_{xxx} at the center is given by f_2 at $z = 1$, which also can be determined as a limit of a series

Extremely slow convergence, but he limit can be very precisely calculated using high order Richardson extrapolation
– this requires several digits of precision floating point calculations

Solution of the differential equation is reduced to the summation of a convergent series $\longrightarrow$ "exact solution"

Calculate the first ~ 1000 coefficients c_n to ~ 1000 digits precision, then use the last ~ 500 terms for Richardson extrapolation
– can get ~ 100 digits precision for y_{xxx}

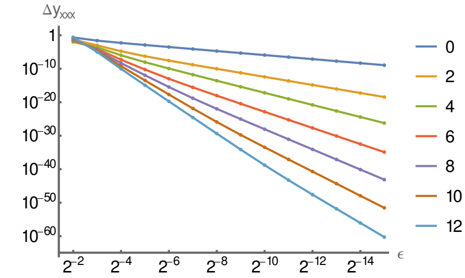
The WKB method connects the tail amplitude α_m of the symmetric solution to the central y_{xxx} of the asymmetric solution
– we get ϵ-expansion result for y_{xxx}



Plot of the third derivative y_{xxx} at the center $x = 0$ as function of ϵ

– at $\epsilon = 2^{-15} = 1/32768$

$$y_{xxx} = 9.3 \cdot 10^{-44684}$$



Relative difference of n-th order ϵ-expansion result $y_{xxx}^{(n)}$ and y_{xxx} obtained by the Hammersley method

$$\Delta y_{xxx} = \frac{y_{xxx}^{(n)} - y_{xxx}}{y_{xxx}}$$

– power law decrease

Conclusions and things to do

How to calculate minimal tail-amplitude?

For smallest ϵ values

$$\alpha_m = \frac{\pi K}{\epsilon^2} \exp\left(-\frac{k\pi}{2\gamma\epsilon}\right)\left(1 - \xi_2\gamma^2\epsilon^2 - \xi_4\gamma^4\epsilon^4 - \xi_6\gamma^6\epsilon^6 - \ldots\right)$$

known up to ξ_{12} , relative error $\sim \epsilon^{14}$

Moderate ϵ values: Hammersley formalism

u_{xxx} is easy to calculate to hundreds of digits

but WKB solves linearized problem $\longrightarrow$ error is $\sim \alpha_m^2$

– if $\alpha_m \approx 10^{-n}$ then we can get n digits precision

$1 > \epsilon > 0.1$: spectral numerical method

we match to linearized tail $\longrightarrow$ error is also $\sim \alpha_m^2$

Scalar field oscillons...