Order of the chiral phase transition for N_f flavors

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GF and T. Hatsuda, arXiv:2404.00554
GF, Phys. Rev. D**105**, L071506 (2022)

Introduction

- QCD Lagrangian:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \bar{q}_i \big(i\gamma^\mu (D)_{ij} - m\delta_{ij} \big) q_j$$

$\longrightarrow$ $SU(3)$ gauge symmetry
$\longrightarrow$ $U_L(N_f) \times U_R(N_f)$ global (approx.) chiral symmetry
$\longrightarrow$ anomalous breaking of $U_A(1)$ axial symmetry

- At low temperatures: spontaneous breaking $SU_L(N_f) \times SU_R(N_f) \longrightarrow SU_V(N_f)$

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- At low temperatures: spontaneous breaking
$SU_L(N_f) \times SU_R(N_f) \longrightarrow SU_V(N_f)$
- Ginzburg-Landau paradigm for second order
(or weakly first order) transitions:

i.) there exists a local order parameter Φ near the transition
ii.) the free energy can be expanded in terms of Φ
iii.) structure of the free energy $\longleftrightarrow$ symmetries

Ginzburg–Landau analysis of the chiral transition

- GL theory for the chiral transition:
  $\longrightarrow$ gauge degrees of freedom are integrated out
  $\longrightarrow$ the emerging order parameter (Φ) is a $N_f \times N_f$ matrix corresponding to $\bar{q}^i_L q^j_R$
  $\longrightarrow$ chiral transformation: $\Phi \to L\Phi R^\dagger$
- The most general free energy functional (no anomaly):

$$\Gamma = \int_x \Big[m^2 \,\mathrm{Tr}\,(\Phi^\dagger\Phi) + g_1 \big(\mathrm{Tr}\,(\Phi^\dagger\Phi)\big)^2 + g_2\,\mathrm{Tr}\,(\Phi^\dagger\Phi\Phi^\dagger\Phi) + \dots + \mathrm{Tr}\,(\partial_i\Phi^\dagger\partial_i\Phi) + \dots\Big]$$

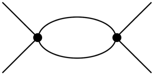
- $U_A(1)$ anomaly: Kobayashi–Maskawa–'t Hooft determinant
  $\longrightarrow$ $a\,(\det\Phi^\dagger + \det\Phi)$
- Free energy is non-analytic at the critical point
  $\longrightarrow$ at T_C long wavelength fluctuations are important
  $\longrightarrow$ the UV free energy is analytic, expansion justified

Ginzburg–Landau analysis of the chiral transition

- Pisarski & Wilczek analysis of the Ginzburg–Landau theory [1]:

  $\longrightarrow$ one-loop calculation of the β functions (no anomaly)

  $\longrightarrow$ counterterms for g_1, g_2:

$\delta g_1, \delta g_2 \sim$ 

- Results (ϵ-expansion, $\epsilon = 4 - d$):

$$\beta_{g_1} = -\epsilon g_1 + \frac{N_f^2 + 4}{4\pi^2} g_1^2 + \frac{N_f}{\pi^2} g_1 g_2 + \frac{3 g_2^2}{4\pi^2}$$

$$\beta_{g_2} = -\epsilon g_2 + \frac{3}{2\pi^2} g_1 g_2 + \frac{N_f}{2\pi^2} g_2^2$$

- No infrared stable fixed point at T_C if $N_f > \sqrt{3}$

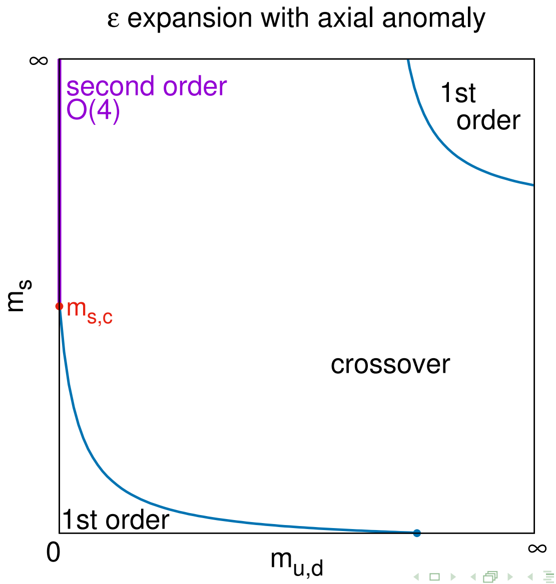
  $\Longrightarrow$ 2nd order transition cannot occur!

- Inclusion of the anomaly: might be 2nd order for $N_f = 2$

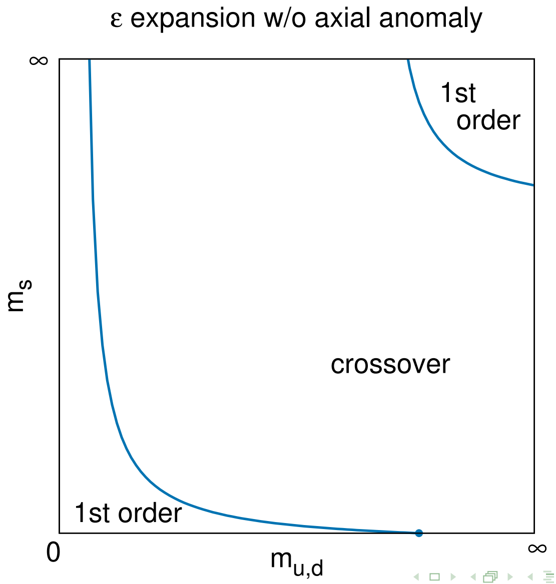
  [$O(4)$ exponents]

[1] R. D. Pisarski and F. Wilczek, Phys. Rev. D**29**, 338 (1984)

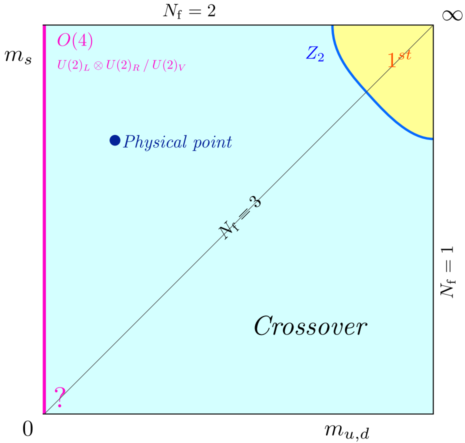
Columbia plot:



Columbia plot:



- Recent lattice QCD result (unimproved staggered fermions):
  $\longrightarrow$ chiral transition is of second order for all N_f up to the conformal window[2]



[2] F. Cuteri, O. Philipsen, and A. Sciarra, JHEP **11**, 141 (2021).

Ginzburg–Landau analysis of the chiral transition

- Lattice QCD result with highly improved staggered fermions ($N_f = 3$)[3]:
  $\longrightarrow$ no direct evidence of a first order transition for
  80 MeV $\lesssim m_\pi \lesssim$ 140 MeV
- Lattice QCD with Mobius domain wall fermions ($N_f = 3$)[4]:
  $\longrightarrow$ critical quark mass $m_{q,\,\mathrm{crit}} \lesssim$ 4 MeV
- Dyson-Schwinger approach[5]:
  $\longrightarrow$ absense of a first order transition for $N_f = 3$
- Non-perturbative conformal bootstrap approach[6]:
  $\longrightarrow$ the transition can be of second order for $N_f = 3$

Contradiction: **where is the corresponding IR fixed point?**

[3] L. Dini et al., Phys. Rev. D105, 034510 (2022)
[4] Y. Zhang et al., arXiv:2401.05066
[5] J. Bernhardt and C.-S. Fischer, Phys. Rev. D**108**,114018 (2023)
[6] S. R. Kousvos and A. Stergiou, SciPost Phys. **15**, 075 (2023)

Ginzburg–Landau analysis of the chiral transition

- Potential problems with the Pisarski & Wilczek analysis:
  $\longrightarrow$ it uses the field theoretical RG
  $\Longrightarrow$ β functions are obtained from UV divergences
  (mass parameter does not appear)
  $\longrightarrow$ number of (perturbatively) relevant operators are restricted at $d \approx 4$
  $\longrightarrow$ $SU(N_f) \times SU(N_f)$ symmetry allows for a richer structure of the free energy in $d = 3$
- Naive scaling: $d = 4$: operators up to $\mathcal{O}(\phi^4)$ are relevant
  $d = 3$: operators up to $\mathcal{O}(\phi^6)$ are relevant
- Results of the ϵ expansion at LO are insensitive to the introduction of higher order terms
  $\longrightarrow$ an inherently $d = 3$ approach is important
  $\longrightarrow$ functional renormalization group (FRG)

Functional Renormalization Group

- FRG generalizes the idea of the WRG: fluctuations are taken into account at the level of the quantum effective action

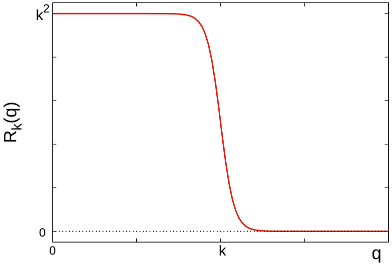
$$Z[J] = \int \mathcal{D}\phi e^{-(\mathcal{S}[\phi] + \int J\phi)} \quad \Rightarrow \quad \Gamma[\bar{\phi}] = -\log Z[J] - \int J\bar{\phi}$$

- Introduction of a flow parameter k and inclusion of fluctuations for which $q \gtrsim k$

$$Z_k[J] = \int \mathcal{D}\phi e^{-(\mathcal{S}[\phi] + \int J\phi)} \times e^{-\frac{1}{2}\int \phi R_k \phi}$$

$\longrightarrow$ regulator: mom. dep. mass term suppressing low modes

$\longrightarrow$ take the $k \to 0$ limit



Functional Renormalization Group

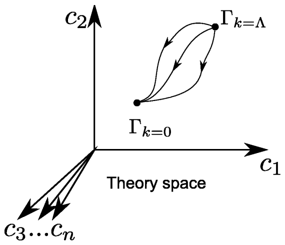
- Scale-dependent effective action:

$$\Gamma_k[\bar\phi] \quad = \quad -\log Z_k[J] - \int J\bar\phi - \frac{1}{2}\int \bar\phi R_k \bar\phi$$

$\longrightarrow k \approx \Lambda$: no fluctuations $\quad \Rightarrow \Gamma_{k=\Lambda}[\bar\phi] = \mathcal{S}[\bar\phi]$

$\longrightarrow k = 0$: all fluctuations $\quad \Rightarrow \Gamma_{k=0}[\bar\phi] = \Gamma[\bar\phi]$

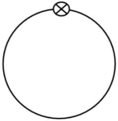
- The scale-dependent effective action interpolates between classical- and quantum effective actions
- The trajectory depends on R_k but the endpoint does not
- Choice of $R_k \leftrightarrow$ optimization!



Functional Renormalization Group

- Flow of the effective action is described by the Wetterich equation:

$$\partial_k \Gamma_k = \frac{1}{2}\int_q \int_p \mathrm{Tr}\,[\partial_k R_k(q,p)(\Gamma_k^{(2)} + R_k)^{-1}(p,q)] = \frac{1}{2}$$



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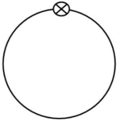
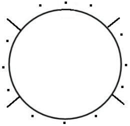


- Slightly different form: [$\tilde{\partial}_k$ acts only on R_k]

$$\partial_k \Gamma_k = \frac{1}{2}\int \tilde{\partial}_k \,\mathrm{Tr}\,\log[\Gamma_k^{(2)} + R_k] = \frac{1}{2}\tilde{\partial}_k \sum$$



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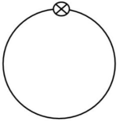
$$\partial_k \Gamma_k = \frac{1}{2}\int_q \int_p \mathrm{Tr}\,[\partial_k R_k(q,p)(\Gamma_k^{(2)} + R_k)^{-1}(p,q)] = \frac{1}{2}$$ 

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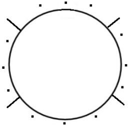
$$\partial_k \Gamma_k = \frac{1}{2}\int \tilde{\partial}_k\, \mathrm{Tr}\, \log[\Gamma_k^{(2)} + R_k] = \frac{1}{2}\tilde{\partial}_k \sum$$ 

- One-loop structure:
  $\longrightarrow$ RG change in the n-point vertices are described by one-loop diagrams [propagators are dressed!]
  $\longrightarrow$ functional integro-differential equation [exact!]

- Main advantage: flows are directly accessible in any dimension
  but approximation is needed

Functional Renormalization Group

- Local potential approximation (LPA):

$$\Gamma_k[\Phi] = \int_x \left(\frac{1}{2}\,\mathrm{Tr}\,[\partial_i\Phi^\dagger\partial_i\Phi] + V_k(\Phi)\right)$$

$\longrightarrow$ $\mathcal{O}(\partial^2)$ of the derivative expansion

$\longrightarrow$ equivalent statement: momentum dependence only in $\Gamma_k^{(2)}$

- No small parameter, optimization important!

- Optimal regulator:

$$R_k(q) = (k^2 - q^2)\Theta(k^2 - q^2)$$

$\longrightarrow$ derivative expansion does converge[7]

[7] I. Balog et al., Phys. Rev. Lett. **123**, 240604 (2019)

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- Optimal flow equation for the effective potential:

$$k\partial_k V_k \quad = \quad \frac{k^5}{6\pi^2}\,\mathrm{Tr}\,(k^2 + V_k^{(2)})^{-1}$$

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Chiral transition with the FRG

- How to build up the most general Ginzburg–Landau potential for N_f flavors in $d = 3$ in terms of renormalizable operators?
  $\longrightarrow$ renormalizable $\equiv$ perturbatively relevant (or marginal)
- Dimension of a scalar field in d dimensions: $[\phi] = (2-d)/2$
  $\longrightarrow$ coupling dimension for $\sim g_n\phi^n$: $[g_n] = ((2-d)n + 2d)/2$
  $\longrightarrow$ for $d = 3$ we need $\mathcal{O}(\phi^6)$!
- Independent invariant for N_f flavors:

$$\begin{aligned} I_1 &= \mathrm{Tr}\,[\Phi^\dagger\Phi] \\ I_2 &= \mathrm{Tr}\,[\Phi^\dagger\Phi\Phi^\dagger\Phi] \\ I_3 &= \mathrm{Tr}\,[\Phi^\dagger\Phi\Phi^\dagger\Phi\Phi^\dagger\Phi] \\ &\dots \\ I_{N_f} &= \mathrm{Tr}\,[(\Phi^\dagger\Phi)^{N_f}] \end{aligned}$$

$\longrightarrow$ only I_1, I_2 and I_3 enters to the potential
(for $N_f = 2$, I_3 is not independent)

Chiral transition with the FRG

- The most general chirally symmetric renormalizable potential:

$$\begin{aligned} V_{ch}[\Phi] &= m^2 \,\mathrm{Tr}\,[\Phi^\dagger\Phi] + g_1\left(\mathrm{Tr}\,[\Phi^\dagger\Phi]\right)^2 + g_2\,\mathrm{Tr}\,[\Phi^\dagger\Phi\Phi^\dagger\Phi] \\ &+ \lambda_1\left(\mathrm{Tr}\,[\Phi^\dagger\Phi]\right)^3 + \lambda_2\,\mathrm{Tr}\,[\Phi^\dagger\Phi]\cdot\,\mathrm{Tr}\,[\Phi^\dagger\Phi\Phi^\dagger\Phi] \\ &+ g_3\,\mathrm{Tr}\,[\Phi^\dagger\Phi\Phi^\dagger\Phi\Phi^\dagger\Phi] \end{aligned}$$

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- Possible $U_A(1)$ breaking terms:

$$I_{\mathrm{det}} = \det\Phi^\dagger + \det\Phi, \qquad \cancel{\tilde{I}_{\mathrm{det}} = \det\Phi^\dagger - \det\Phi}$$

$\longrightarrow$ $\tilde{I}^2_{\mathrm{det}}$ and $\det\Phi^\dagger\cdot\det\Phi$ are not independent from I_{det} and the I_i

- If Φ is too large, I_{det} becomes perturbatively irrelevant!

$\longrightarrow$ $I_{\mathrm{det}} \sim \mathcal{O}(\phi^6)$

- For $N_f > 6$ the potential does not contain the anomaly

Chiral transition with the FRG

- $\mathbf{N_f = 5, 6}$:

  $$V_A = a \cdot (\det \Phi^\dagger + \det \Phi)$$

Chiral transition with the FRG

- $\mathbf{N_f = 5, 6}$:

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- $\mathbf{N_f = 4}$:

$$V_A = a \cdot (\det \Phi^\dagger + \det \Phi) + b \cdot \mathrm{Tr}\,[\Phi^\dagger \Phi](\det \Phi^\dagger + \det \Phi)$$

Chiral transition with the FRG

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- **$N_f = 3$**:

$$V_A = a \cdot (\det \Phi^\dagger + \det \Phi) + b \cdot \mathrm{Tr}\,[\Phi^\dagger \Phi](\det \Phi^\dagger + \det \Phi) + a_2 \cdot (\det \Phi^\dagger + \det \Phi)^2$$

Chiral transition with the FRG

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- $\mathbf{N_f = 2}$:

$$\begin{aligned} V_A = {} & a \cdot (\det \Phi^\dagger + \det \Phi) + b_1 \cdot \mathrm{Tr}\,[\Phi^\dagger \Phi](\det \Phi^\dagger + \det \Phi) \\ & + a_2 \cdot (\det \Phi^\dagger + \det \Phi)^2 + a_3 \cdot (\det \Phi^\dagger + \det \Phi)^3 \\ & + b_2 \cdot \left(\mathrm{Tr}\,[\Phi^\dagger \Phi]\right)^2 (\det \Phi^\dagger + \det \Phi) \\ & + b_3 \cdot \left(\mathrm{Tr}\,[\Phi^\dagger \Phi]\right)^3 (\det \Phi^\dagger + \det \Phi) + b_4 \cdot \mathrm{Tr}\,(\Phi^\dagger \Phi)^2 (\det \Phi^\dagger + \det \Phi) \end{aligned}$$

Chiral transition with the FRG

- Optimized flow equation:

$$k\partial_k V_k = \frac{k^5}{6\pi^2}\,\mathrm{Tr}\,[k^2 + V_k^{(2)}]^{-1}$$

- Identification of the scale dependencies:

$$\sum_n k\partial_k g_n \cdot \mathcal{O}_n = \sum_n \frac{k^5}{6\pi^2}[\ldots] \cdot \mathcal{O}_n$$

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- Problem:

  $\longrightarrow$ $V_k^{(2)}$ depends on the fields, not invariants!

  $\longrightarrow$ $[k^2 + V_k^{(2)}]$: $2N_f^2 \times 2N_f^2$ matrix, in practice cannot be inverted for a general field configuration

- Specific background:

$$\Phi = s_0 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & ... & \\ & & & 1 \end{pmatrix} + s_L \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & ... & \\ & & & -(N_f - 1) \end{pmatrix}$$

Chiral transition with the FRG

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- Identification of the scale dependencies:

$$\sum_n k\partial_k g_n \cdot \mathcal{O}_n = \sum_n \frac{k^5}{6\pi^2}[\ldots]\cdot \mathcal{O}_n$$

- The $\mathcal{O}_n$ operators become linear combinations:

$$\mathcal{O}_n = \sum_{\alpha+\beta=n} c^{\alpha\beta} s_0^\alpha s_L^\beta$$

$\longrightarrow$ at each order matching *rhs* and *lhs* leads to coupling flows

- β functions: $(g_n = k^{(6-n)/2}\bar{g}_n)$

$$\beta_n \equiv k\partial_k \bar{g}_n = -\frac{1}{2}(6-n)\bar{g}_n + k\partial_k g_n / k^{(6-n)/2}$$

- β functions without anomaly:

$$
\begin{aligned}
\beta_{m^2} &= -2\bar{m}_k^2 - 2\frac{\bar{g}_{1,k}N_f(N_f^2+1)+\bar{g}_{2,k}(N_f^2-1)}{3\pi^2 N_f(1+\bar{m}_k^2)^2},\\
\beta_{g_1} &= -\bar{g}_{1,k} + 4\frac{\bar{g}_{1,k}^2N_f^2(N_f^2+4)+2\bar{g}_{1,k}\bar{g}_{2,k}N_f(N_f^2-1)+2\bar{g}_{2,k}^2(N_f^2-1)}{3\pi^2N_f^2(1+\bar{m}_k^2)^3} - \frac{3\bar{\lambda}_{1,k}N_f(N_f^2+2)+2\bar{\lambda}_{2,k}(N_f^2-1)}{3\pi^2N_f(1+\bar{m}_k^2)^2},\\
\beta_{g_2} &= -\bar{g}_{2,k} + 8\frac{3\bar{g}_{1,k}\bar{g}_{2,k}N_f+\bar{g}_{2,k}^2(N_f^2-3)}{3\pi^2N_f(1+\bar{m}_k^2)^3} - \frac{3\bar{g}_{3,k}(N_f^2-4)+\bar{\lambda}_{2,k}N_f(N_f^2+4)}{3\pi^2N_f(1+\bar{m}_k^2)^2},\\
\beta_{\lambda_1} &= 4\frac{\bar{g}_{1,k}N_f^2(3\bar{\lambda}_{1,k}N_f(N_f^2+7)+2\bar{\lambda}_{2,k}(N_f^2-1))+\bar{g}_{2,k}N_f(N_f^2-1)(3N_f\bar{\lambda}_{1,k}+4\bar{\lambda}_{2,k})}{3\pi^2N_f^3(1+\bar{m}_k^2)^3}\\
&- 4\frac{2\bar{g}_{1,k}^3N_f^3(N_f^2+13)+6\bar{g}_{1,k}^2\bar{g}_{2,k}N_f^2(N_f^2-1)+12\bar{g}_{1,k}\bar{g}_{2,k}^2N_f(N_f^2-1)+8\bar{g}_{2,k}^3(N_f^2-1)}{3\pi^2N_f^3(1+\bar{m}_k^2)^4},\\
\beta_{\lambda_2} &= 4\frac{\bar{g}_{1,k}N_f\left(\bar{\lambda}_{2,k}N_f(N_f^2+19)+3\bar{g}_{3,k}(N_f^2-4)\right)+\bar{g}_{2,k}\left(15\bar{g}_{3,k}(N_f^2-4)+N_f(18\bar{\lambda}_{1,k}N_f+\bar{\lambda}_{2,k}(5N_f^2-1))\right)}{3\pi^2N_f^2(1+\bar{m}_k^2)^3}\\
&- 4\frac{72N_f^2\bar{g}_{1,k}^2\bar{g}_{2,k}+6\bar{g}_{1,k}\bar{g}_{2,k}^2N_f(2N_f^2+3)+\bar{g}_{2,k}^3(24N_f^2-90)}{3\pi^2N_f^2(1+\bar{m}_k^2)^4},\\
\beta_{g_3} &= 4\frac{5N_f\bar{g}_{1,k}\bar{g}_{3,k}+4N_f\bar{g}_{2,k}\bar{\lambda}_{2,k}+(2N_f^2-17)\bar{g}_{2,k}\bar{g}_{3,k}}{\pi^2N_f(1+\bar{m}_k^2)^3} - 4\frac{54\bar{g}_{1,k}\bar{g}_{2,k}^2N_f+\bar{g}_{2,k}^3(4N_f^2-54)}{3\pi^2N_f(1+\bar{m}_k^2)^4}.
\end{aligned}
$$

- Fixed points: $\beta_i = 0 \ \forall \ i$

  $\longrightarrow$ solve for marginal couplings

  $\longrightarrow$ substitute to the relevant couplings

  $\longrightarrow$ find fixed points

  $\longrightarrow$ check stability matrix $(\partial\beta_i/\partial g_j)$ at fixed points

$\mathbf{N_f}$	FP	$\bar{m}^2$	$\bar{g}_1$	$\bar{g}_2$	RD#
50	$O(2N_f^2)$	−0.33342	0.0017538	0	2
"	B_2^{50}	0.040303	−0.0029448	0.12152	2
"	C_1^{50}	−0.37509	0.0019579	−0.011198	1
"	$\tilde{C}_1^{50}$	−0.33342	0.0017556	−0.000088291	1
20	$O(2N_f^2)$	−0.33385	0.010939	0	2
"	B_2^{20}	0.043192	−0.018915	0.31043	2
"	C_1^{20}	−0.38411	0.012287	−0.030728	1
"	$\tilde{C}_1^{20}$	−0.33393	0.011010	−0.0014253	1
10	$O(2N_f^2)$	−0.33492	0.043430	0	2
"	B_2^{10}	0.059163	−0.086421	0.68317	2
"	C_1^{10}	−0.43356	0.048876	−0.082581	1
"	$\tilde{C}_1^{10}$	−0.33641	0.044669	−0.012667	1
6	$O(2N_f^2)$	−0.33516	0.11855	0	2
"	B_2^{6}	0.40276	−1.23414	3.80527	2
"	C_1^{6}	1.09084	−6.45942	16.76628	1
"	$\tilde{C}_1^{6}$	−0.34848	0.12934	−0.069536	1

$\mathbf{N_f}$	FP	$\bar{m}^2$	$\bar{g}_1$	$\bar{g}_2$	$\bar{a}$	RD#
5	$O(2N_f^2)$	-0.33386	0.16871	0	0	2
''	$\tilde{C}_1^5$	-0.36068	0.19128	-0.12675	0	1
''	A_3^5	-0.17023	0.14387	-0.056313	-2.79735	3

$\mathbf{N_f}$	FP	$\bar{m}^2$	$\bar{g}_1$	$\bar{g}_2$	$\bar{a}$	RD#
4	$O(2N_f^2)$	-0.32940	0.25800	0	0	3 (2)
''	$\tilde{C}_2^4$	-0.38129	0.31042	-0.25480	0	2 (1)
''	A_2^4	-0.34949	0.63992	-1.73326	-3.82052	2
''	$\tilde{A}_2^4$	-0.40273	0.21168	0.17473	-0.73657	2

$\mathbf{N_f}$	FP	$\bar{m}^2$	$\bar{g}_1$	$\bar{g}_2$	$\bar{a}$	$\bar{b}$	RD#
3	$O(2N_f^2)$	-0.31496	0.43763	0	0	0	3 (2)
''	$\tilde{C}_2^3$	-0.38262	0.59725	-0.62042	0	0	2 (1)
''	A_4^3	-0.01786	0.091631	-0.14148	-0.11900	0.39087	4
''	A_{1*}^3	-0.41126	0.73099	-0.88199	-0.46585	-0.91131	$1*$

Fixed points and stability

- Anomaly free fixed points for $N_f = 2$:

$\mathbf{N_f}$	FP	$\bar{m}^2$	$\bar{g}_1$	$\bar{g}_2$	RD#
2	$O(2N_f^2)$	-0.27094	0.85280	0	4 (3)
"	$\tilde{C}_2^2$	-0.20599	1.33367	-1.88211	2 (1)
"	$\hat{C}_2^2$	-0.26318	0.33093	1.71728	2 (1)

- Anomalous fixed points? $\longrightarrow$ numerically challenging
  $\longrightarrow |a| = \infty$, $m^2 = \infty$ with $m^2 + a =$ finite $\Rightarrow$ $O(4)$ FP
- For $N_f \geq 5$ the fixed point structure is consistent with a second order phase transition
  $\longrightarrow$ the $U_A(1)$ anomaly does not play any role
  $\longrightarrow$ if $U_A(1)$ is broken at T_c, fluctuations wash out its effect
- For $N_f = 2, 3, 4$ the situation is more subtle

Fixed points and stability

- **Case I.** (flavor continuity)

  $\longrightarrow$ the chiral transition is governed by the $\tilde{C}^{N_f}$ fixed points

  $\longrightarrow$ for $N_f \geq 5$, irrespectively of the $U_A(1)$ anomaly,
  they are IR stable at T_C
  $\Rightarrow$ second order transition

  $\longrightarrow$ for $N_f = 2, 3, 4$, if the $U_A(1)$ anomaly disappears,
  they are IR stable at T_C
  $\Rightarrow$ second order transition

  $\longrightarrow$ it is unlikely that the anomaly *exactly* disappears
  $\Rightarrow$ first order transition for $N_f = 2, 3, 4$,
  which can become weak if $U_A(1)$ breaking is small

Fixed points and stability

- **Case II.**

  $\longrightarrow$ the chiral transition is governed by the $\tilde{C}^{N_f}$ fixed points, except for $N_f = 2$

  $\Rightarrow$ $O(4)$ fixed point is IR stable for strong anomaly at T_C

  $\longrightarrow$ the transition is second order for $N_f \geq 5$, first order for $N_f = 3, 4$, and second order for $N_f = 2$

- **Case III.**

  $\longrightarrow$ the chiral transition is governed by the $\tilde{C}^{N_f}$ fixed points, except for $N_f = 2$ and $N_f = 3$

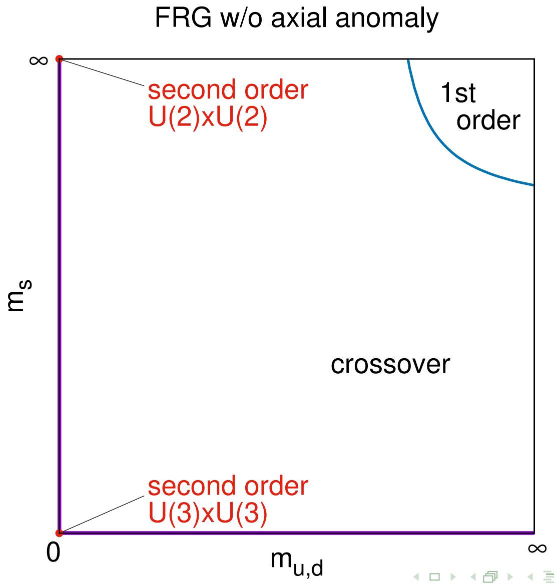
  $\Rightarrow$ $O(4)$ fixed point is IR stable for strong anomaly at T_C

  $\Rightarrow$ A^3_{1*} fixed point is IR stable for nonzero anomaly at T_C

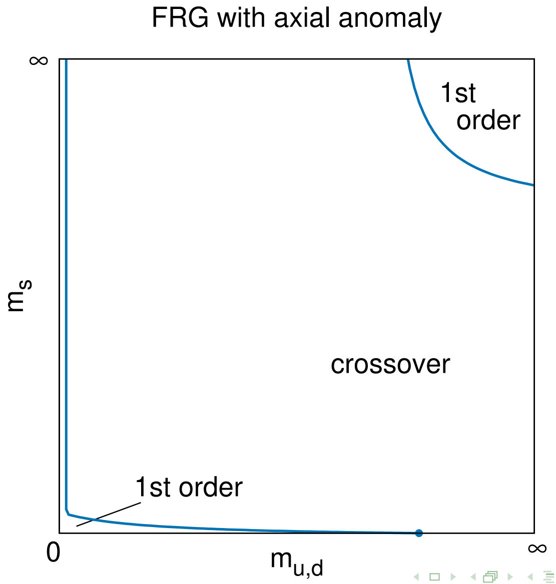
  [not all stability eigenvalues are real!]

  $\longrightarrow$ the transition is second order for $N_f \geq 5$, first order for $N_f = 4$, and second order for $N_f = 2, 3$

Columbia plot:



Columbia plot:



Fixed points and stability

Transition orders without anomaly:

	$N_f = 2$	$N_f = 3$	$N_f = 4$	$N_f \geq 5$
ϵ expansion ($\epsilon = 1$)	1st order	1st order	1st order	1st order
FRG ($d = 3$)	2nd order	2nd order	2nd order	2nd order

Transition orders with anomaly:

	$N_f = 2$	$N_f = 3$	$N_f = 4$	$N_f \geq 5$
ϵ expansion ($\epsilon = 1$)	2nd order*	1st order	1st order	1st order
FRG ($d = 3$)	1st order (Case I) 2nd order (Case II) 2nd order (Case III)	1st order (Case I) 1st order (Case II) 2nd order (Case III)	1st order	2nd order

*:only with strong anomaly

Summary

- Re-analysis of the RG flows of the Ginzburg-Landau potential of chiral transition

  $\longrightarrow$ scale evolution is obtained directly at $d = 3$ using the Functional Renormalization Group method

  $\longrightarrow$ Local Potential Approximation + $\mathcal{O}(\phi^6)$ truncation: including all relevant and marginal interactions

- Results can be made consistent with recent lattice QCD simulations [i.e. chiral transition is second order]

  $\longrightarrow$ there exist new classes of fixed points spanned in the entire N_f range

  $\longrightarrow$ they are IR stable at T_C for $N_f \geq 5$

  $\longrightarrow$ they are IR stable at T_C for $N_f = 2, 3, 4$ only if $U_A(1)$ is restored

- Future:

  $\longrightarrow$ improve truncation (irrelevant operators, wavefunction renormalization, higher derivatives)

  $\longrightarrow$ establishing fully non-perturbative fixed point potentials