

Thermal fate of the $U_A(1)$ anomaly from the functional renormalization group

Gergely Fejős

Eötvös University, Institute of Physics
Dept. of Atomic Physics

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GF and A. Patkos, Phys. Rev. D**105**, 096007 (2022)

GF, Phys. Rev. D**105**, L071506 (2022)

Outline

Introduction

Anomaly evolution in the three flavor $L\sigma M$

Renormalization group analysis of the chiral transition

Summary

Introduction

- QCD Lagrangian:

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} + \bar{q}_i (i\gamma^\mu (D_\mu)_{ij} - m\delta_{ij}) q_j$$

- $SU(3)$ gauge symmetry
- $U_L(N_f) \times U_R(N_f)$ global (approx.) chiral symmetry
- anomalous breaking of $U_A(1)$ axial symmetry
- At low temperatures: spontaneous breaking
 $SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$

Introduction

- QCD Lagrangian:

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- anomalous breaking of $U_A(1)$ axial symmetry
- At low temperatures: spontaneous breaking
 $SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$
- At high temperatures: $U_A(1)$ anomaly disappears
 - semi-classical approximation of instanton density:
[r - instanton size]

$$n(r) \sim \exp[-8(\pi r T)^2/3]$$

- only valid for $T \gg T_c$
- what is the fate of the anomaly for low temperatures?

Anomaly evolution in the three flavor meson model

3 FLAVOR CHIRAL MESON MODEL:

- Low energy effective model: M - meson fields
[excitations of M : π, K, η, η' and a_0, κ, f_0, σ]

$$M = (s^a + i\pi^a) T^a \quad \text{Tr}(T^a T^b) = \delta_{ab}/2$$

- Lagrangian with renormalizable operators (3 + 1 dim.):
(Euclidean!)

$$\begin{aligned} \mathcal{L} = & \text{Tr} [\partial_i M^\dagger \partial_i M] + m^2 \text{Tr} (M^\dagger M) \\ & + g_1 [\text{Tr} (M^\dagger M)]^2 + g_2 \text{Tr} (M^\dagger M M M^\dagger M) \\ & + a (\det M^\dagger + \det M) - \text{Tr} [H(M^\dagger + M)] \end{aligned}$$

- \mathcal{L} contains **renormalizable** operators and it is invariant under **chiral symmetry** [apart from $U_A(1)$]
→ in the quantum effective action **every operator allowed by symmetry** is present!

Anomaly evolution in the three flavor meson model

- Quantum effective action:

$$\Gamma[M] = -\log \int \mathcal{D}\hat{M} \exp \left(- \int_x \mathcal{L}_J[M + \hat{M}] \right) - \int_x \text{Tr}(J^\dagger M + J M^\dagger)$$

→ Ansatz for $\Gamma[M]$? It has to reflect chiral symmetry!

- Chiral invariants for 3 flavors:

$$\rho = \text{Tr}(M^\dagger M),$$

$$\tau = \text{Tr}(M^\dagger M - \rho/3)^2$$

$$\rho_3 = \text{Tr}(M^\dagger M - \rho/3)^3$$

$$\Delta = \det M^\dagger + \det M \rightarrow \text{anomaly!}$$

- Note 1: higher order traces are not independent!
- Note 2: $\tilde{\Delta} \equiv \det M^\dagger - \det M$ has wrong parity
and $\tilde{\Delta}^2$ is not independent!

Anomaly evolution in the three flavor meson model

- Chiral invariants for 3 flavors:

$$\rho = \text{Tr}(\textcolor{blue}{M^\dagger M}),$$

$$\tau = \text{Tr}(\textcolor{blue}{M^\dagger M} - \rho/3)^2$$

$$\rho_3 = \text{Tr}(\textcolor{blue}{M^\dagger M} - \rho/3)^3$$

$$\Delta = \det \textcolor{blue}{M^\dagger} + \det \textcolor{blue}{M} \rightarrow \text{anomaly!}$$

- What terms can describe the anomaly? Infinitely many!

$$\rightarrow \rho\Delta, \rho^2\Delta, \dots, \rho^n\Delta, \dots$$

$$\rightarrow \tau\Delta, \tau^2\Delta, \dots, \tau^n\Delta, \dots \Rightarrow \text{dropped!}$$

$$\rightarrow \rho_3\Delta, \rho_3^2\Delta, \dots, \rho_3^n\Delta, \dots \Rightarrow \text{dropped!}$$

- Chiral limit: in the ground state $M \sim \mathbf{1} \Rightarrow \tau = 0, \rho_3 = 0$!

$$\sum_n \textcolor{red}{a}_n \rho^n \Delta \equiv \textcolor{red}{A}(\rho) \Delta$$

- Higher orders in Δ ? \Rightarrow instantons with $|Q| \neq 1$

Anomaly evolution in the three flavor meson model

- Classical action:

$$S[\mathcal{M}] = \int_x \mathcal{L} = \int_x \left[\text{Tr} [\partial_i \mathcal{M}^\dagger \partial_i \mathcal{M}] + m^2 \rho + \lambda_1 \rho^2 + \lambda_2 \tau + a \Delta - \text{Tr} [H(\mathcal{M}^\dagger + \mathcal{M})] \right]$$

- Quantum effective action (ansatz!):

$$\Gamma[\mathcal{M}] = \int_x \gamma = \int_x \left[\text{Tr} [\partial_i \mathcal{M}^\dagger \partial_i \mathcal{M}] + U(\rho) + C(\rho)\tau + A(\rho)\Delta - \text{Tr} [H(\mathcal{M}^\dagger + \mathcal{M})] \right]$$

- Task: calculate generalized „couplings” $U(\rho)$, $C(\rho)$ and $A(\rho)$!

→ effective tool: **Functional Renormalization Group**

Anomaly evolution in the three flavor meson model

- Scale-dependent effective action:

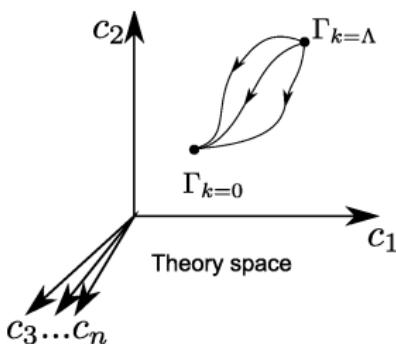
$$\begin{aligned}\Gamma_k[M] = & -\log \int \mathcal{D}\hat{M} \exp \left(- \int_x \mathcal{L}_J[M + \hat{M}] + \int_{xy} \text{Tr}(\hat{M}^\dagger R_k \hat{M}) \right) \\ & - \int_x \text{Tr}(J^\dagger M + J M^\dagger) - \int_{xy} \text{Tr}(M^\dagger R_k M)\end{aligned}$$

→ $k \approx \Lambda$: no fluctuations ⇒ $\Gamma_{k=\Lambda}[M] = \mathcal{S}[M]$

→ $k = 0$: all fluctuations ⇒ $\Gamma_{k=0}[M] = \Gamma[M]$

- The scale-dependent effective action interpolates between classical- and quantum effective actions

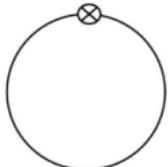
- The trajectory depends on R_k but the endpoint does not



Anomaly evolution in the three flavor meson model

- Wetterich equation:

$$\partial_k \Gamma_k = \frac{1}{2} \int_{qp} \text{Tr} [\partial_k R_k(q, p) (\Gamma_k^{(2)} + R_k)^{-1}(-p, -q)] = \frac{1}{2}$$



- Ansatz for Γ_k :

$$\begin{aligned} \Gamma_k = \int_x \gamma_k &= \int_x \left[\text{Tr} [\partial_i M^\dagger \partial_i M] + V_k(\rho) + C_k(\rho)\tau + A_k(\rho)\Delta \right. \\ &\quad \left. - \text{Tr} [H(M^\dagger + M)] \right] \end{aligned}$$

- Task: project the Wetterich equation onto $\sim \tau$, $\sim \Delta$ operators
 - problem: $\Gamma_k^{(2)}$ is not chirally invariant (but Γ_k has to be!)
 - way out: any background fields can be applied!
- E.g. $M \sim \mathbf{1} \Rightarrow \tau = 0$

Anomaly evolution in the three flavor meson model

- Litim's optimal regulator: $R_k(\omega, \vec{q}) = (k^2 - \vec{q}^2)\Theta(k^2 - \vec{q}^2)$
- Long calculations lead to the following flow equations:

$$\partial_k \textcolor{red}{U}_k(\rho) = \frac{k^3 T}{12\pi^2} \sum_n \tilde{\partial}_k (8 \log D_8 + \log D_0)$$

$$\begin{aligned} \partial_k \textcolor{red}{A}_k(\rho) = & \frac{k^3 T}{6\pi^2} \sum_n \tilde{\partial}_k \left[\frac{8}{D_8} \left(\textcolor{red}{A}'_k (\omega_n^2 + k^2 + \textcolor{red}{U}'_k) + \frac{2}{3} \rho \textcolor{red}{C}_k \textcolor{red}{A}'_k + \textcolor{red}{A}_k \textcolor{red}{C}_k \right) \right. \\ & \left. + \frac{1}{D_0} \left((4\textcolor{red}{A}'_k + \rho \textcolor{red}{A}''_k)(\omega_n^2 + k^2 + \textcolor{red}{U}'_k) + \textcolor{red}{U}''_k (\rho \textcolor{red}{A}'_k - 3\textcolor{red}{A}_k) \right) \right] \end{aligned}$$

$$\partial_k \textcolor{red}{C}_k(\rho) = \frac{k^3 T}{6\pi^2} \sum_n \tilde{\partial}_k [\dots]$$

with

$$D_8 = (\omega_n^2 + k^2 + \textcolor{red}{U}'_k)(\omega_n^2 + k^2 + \textcolor{red}{U}'_k + \frac{4}{3}\rho \textcolor{red}{C}_k) - \frac{1}{3}\rho \textcolor{red}{A}_k^2$$

$$D_0 = (\omega_n^2 + k^2 + \textcolor{red}{U}'_k)(\omega_n^2 + k^2 + \textcolor{red}{U}'_k + 2\rho \textcolor{red}{U}''_k) - \frac{4}{3}\rho (\textcolor{red}{A}_k + \rho \textcolor{red}{A}'_k)^2$$

Anomaly evolution in the three flavor meson model

- Numerical solution: **grid method**
- Start from $k = \Lambda \equiv 1 \text{ GeV}$ and integrate toward $k \rightarrow 0$
($k_{\text{end}} = 10 \text{ MeV}$)

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- Classical action:

$$S = \int_x \left[\text{Tr} [\partial_i M^\dagger \partial_i M] + m^2 \rho + \lambda_1 \rho^2 + \lambda_2 \tau + a \Delta - h_a s^a \right]$$

- Quantum effective action:

$$\Gamma_k = \int_x \left[\text{Tr} [\partial_i M^\dagger \partial_i M] + U_k(\rho) + C_k(\rho) \tau + A_k(\rho) \Delta - h_a s^a \right]$$

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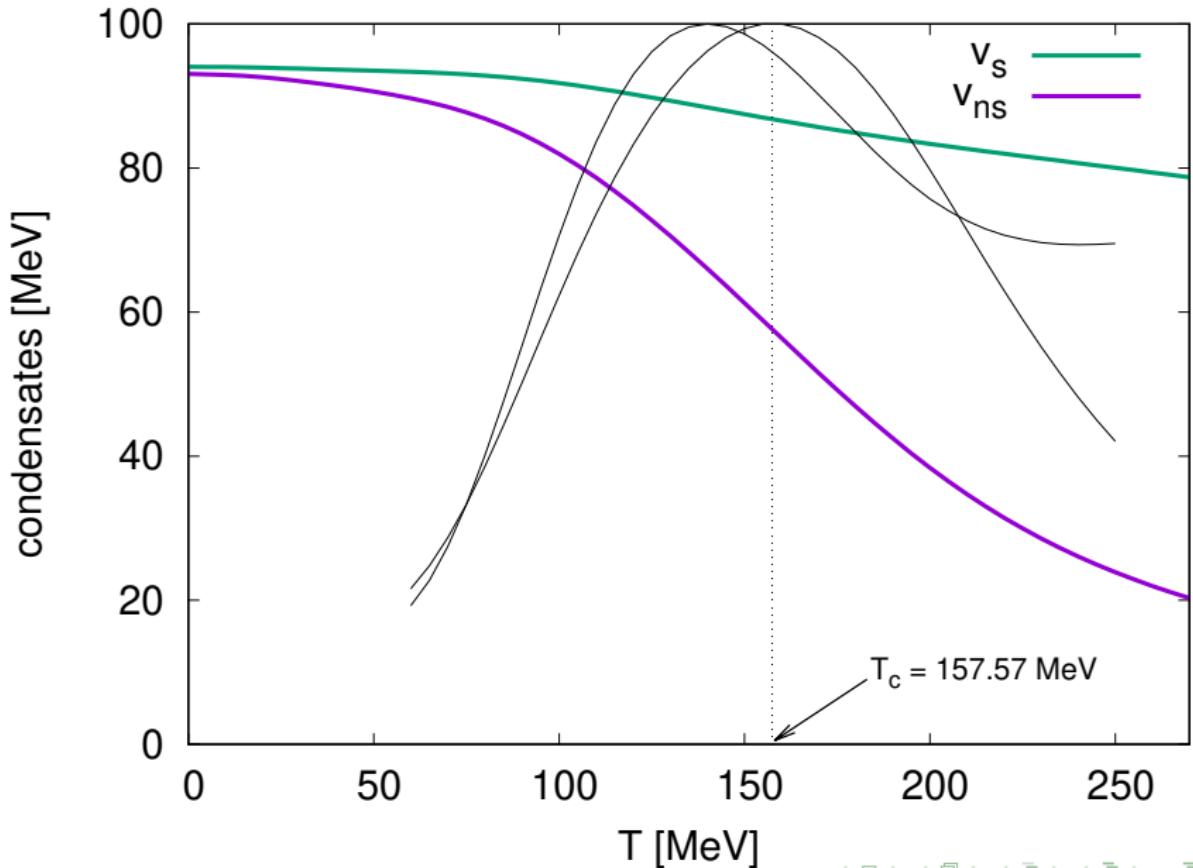
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- Quantum effective action:

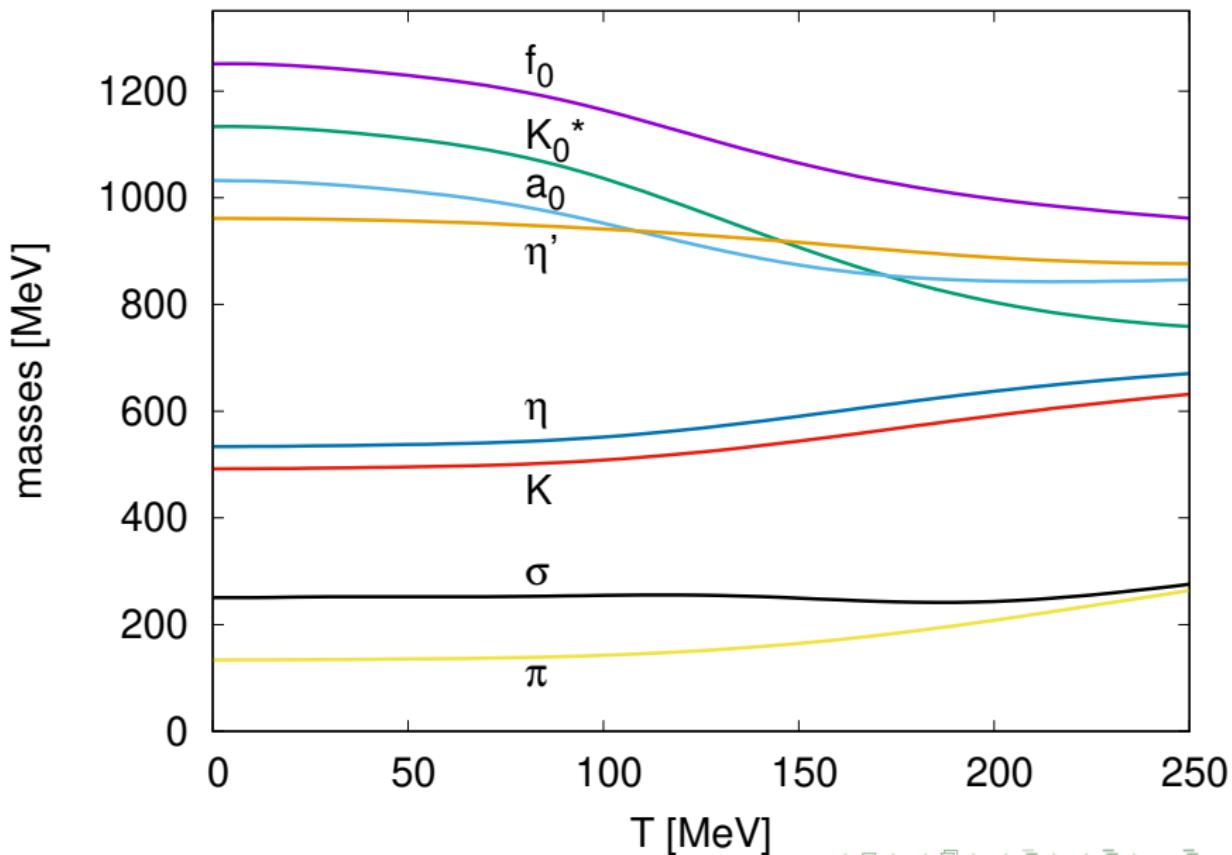
$$\Gamma_k = \int_x \left[\text{Tr} [\partial_i M^\dagger \partial_i M] + U_k(\rho) + C_k(\rho) \tau + A_k(\rho) \Delta - h_a s^a \right]$$

- Six unknown parameters ($m^2, g_1, g_2, a, h_s, h_{ns}$) \Rightarrow six inputs!
- 2 PCAC relations (f_π, f_K decay const.'s), 4 masses (π, K, η, η')

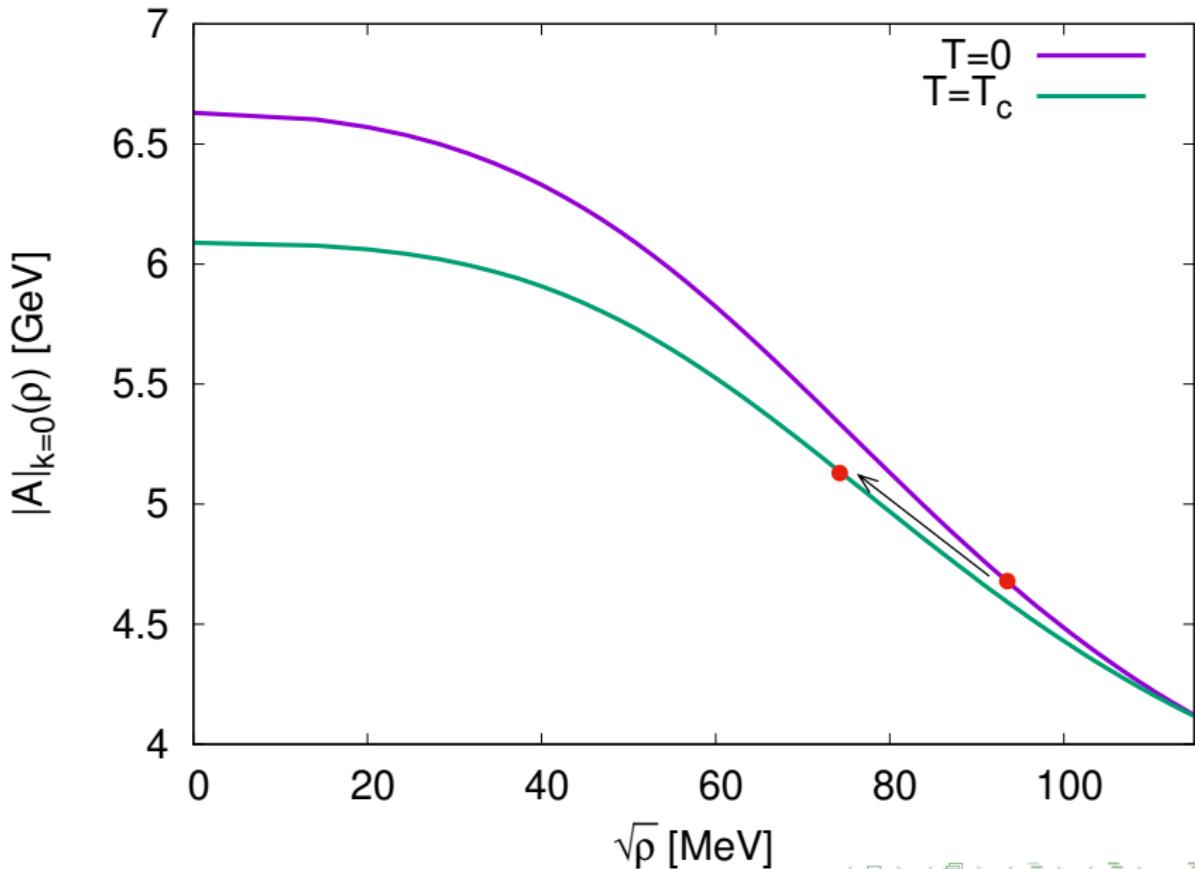
Anomaly evolution in the three flavor meson model



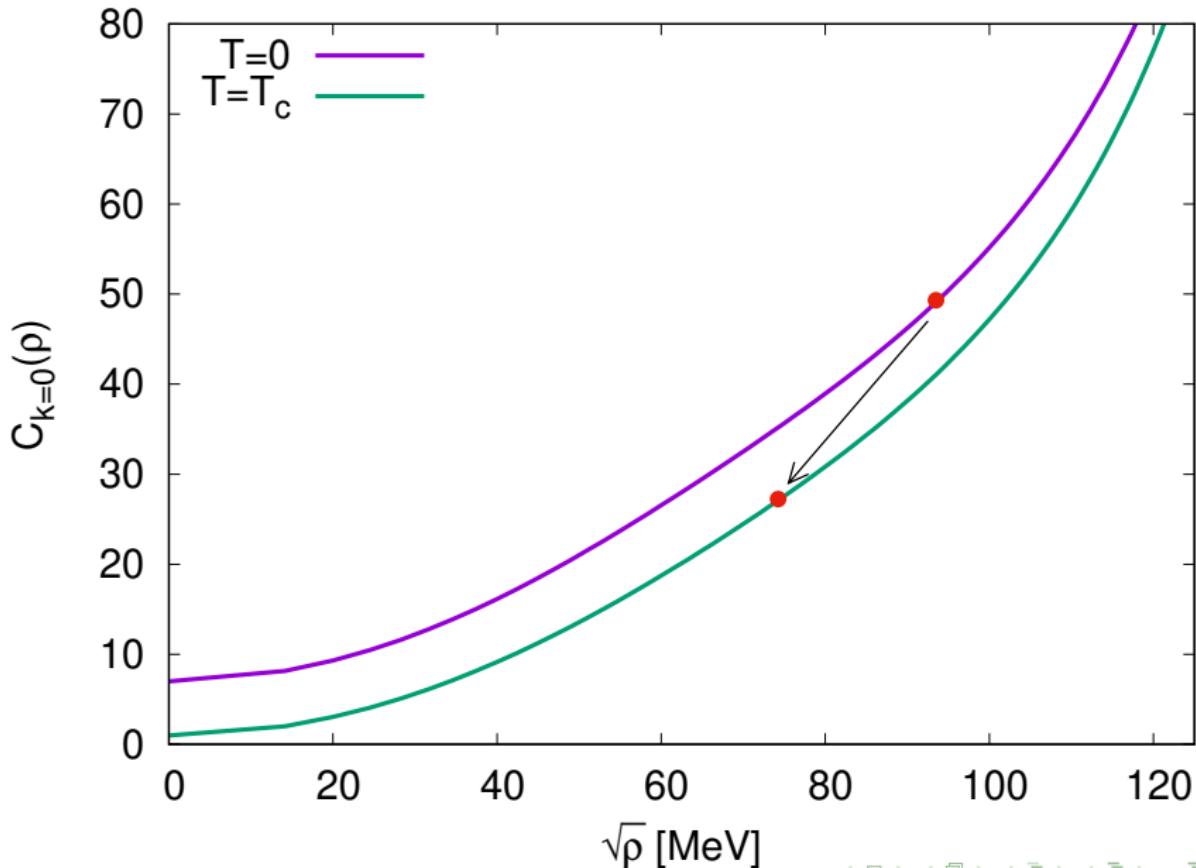
Anomaly evolution in the three flavor meson model



Anomaly evolution in the three flavor meson model



Anomaly evolution in the three flavor meson model



Anomaly evolution in the three flavor meson model

- How can the anomaly grow with T ?
 - instanton effects are not yet included!
 - anomaly strengthening is caused by mesonic fluctuations
- Instantons are hidden in the bare anomaly parameter (a)
 - how to implement topological fluctuations?
 - in principle parameter a is T -dependent

Anomaly evolution in the three flavor meson model

- How can the anomaly grow with T ?
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 - anomaly strengthening is caused by mesonic fluctuations
- Instantons are hidden in the bare anomaly parameter (a)
 - how to implement topological fluctuations?
 - in principle parameter a is T -dependent
- Three additional scenarios:

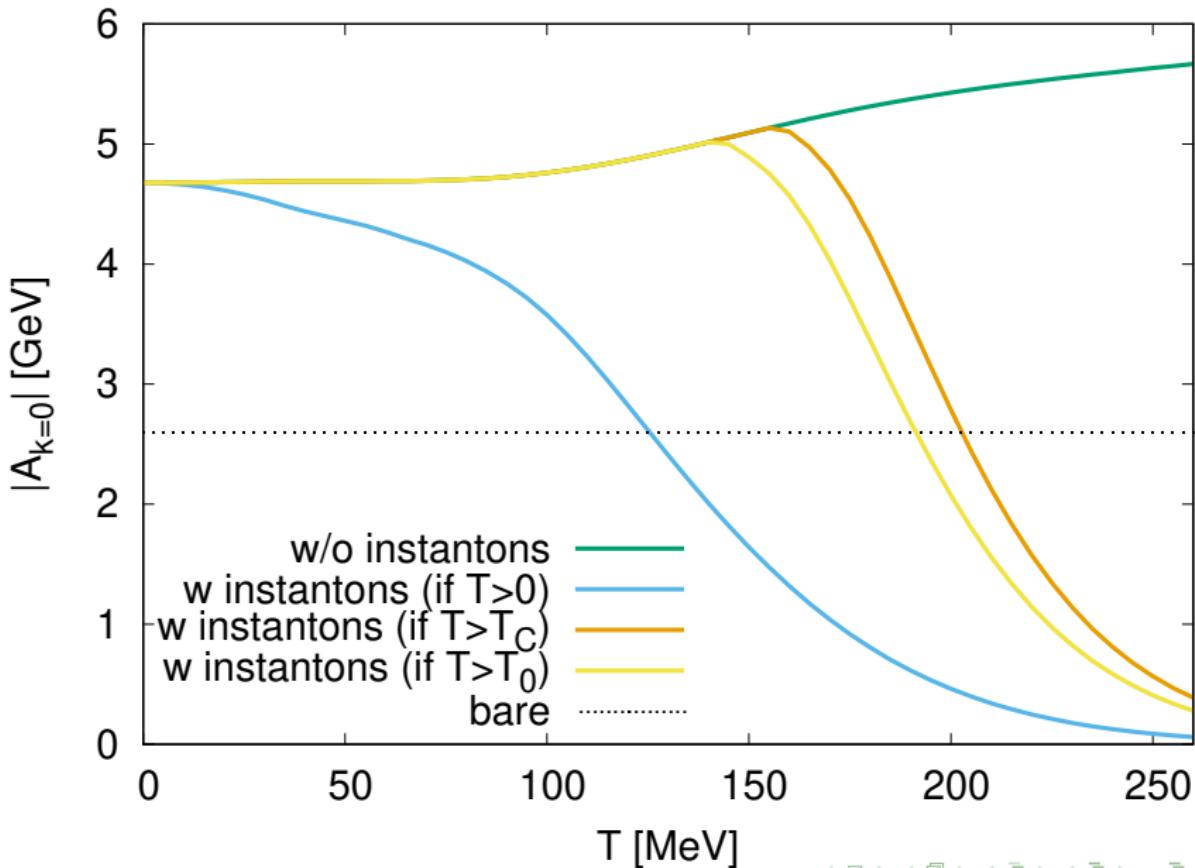
$$i) \quad a(T) = a_0 \exp[-8(\pi r T)^2/3],$$

$$ii) \quad a(T) = \begin{cases} a_0, & \text{if } T < T_c \\ a_0 \exp[-8(\pi r)^2(T^2 - T_c^2)/3], & \text{else} \end{cases}$$

$$iii) \quad a(T) = \begin{cases} a_0, & \text{if } T < T_0 \\ a_0 \exp[-8(\pi r)^2(T^2 - T_0^2)/3], & \text{else.} \end{cases}$$

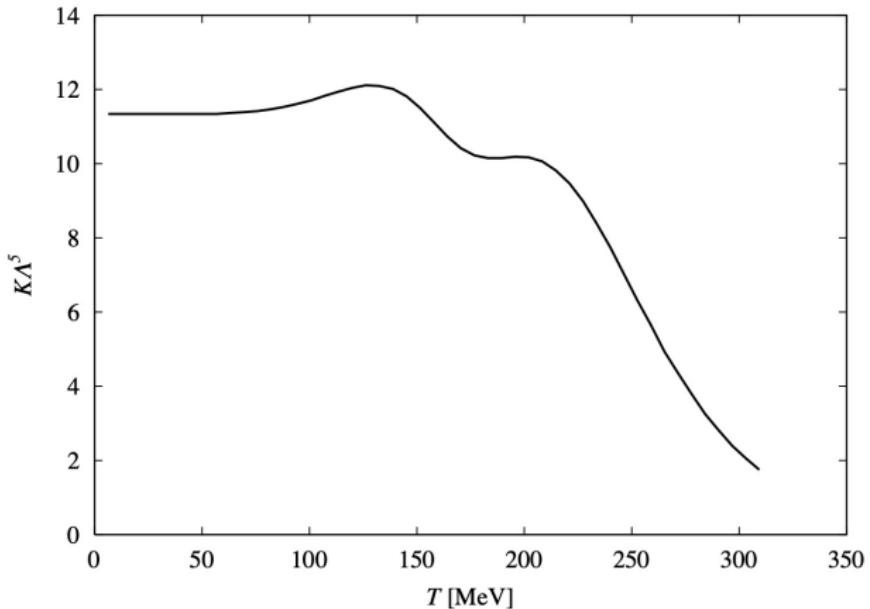
→ $r \simeq 1/3 \text{ fm}$, $T_0 \approx 143 \text{ MeV}$

Anomaly evolution in the three flavor meson model



Anomaly evolution in the three flavor meson model

- Similar behavior in *K. Fukushima et. al*, PRC**63** (2001) 045203
→ NJL model calculation: fitting the **K anomaly parameter**
to recover χ_{top} of the lattice



Renormalization group analysis of the chiral transition

- Results depend on the choice of $a(T)$
 - low T assumption (i.e. $a(T) \approx a_0$) might need more care
 - connection with the **transition order** (in the chiral limit)!

Renormalization group analysis of the chiral transition

- Results depend on the choice of $a(T)$
 - low T assumption (i.e. $a(T) \approx a_0$) might need more care
 - connection with the **transition order** (in the chiral limit)!
- At low temperatures: spontaneous breaking
 $SU_L(N_f) \times SU_R(N_f) \longrightarrow SU_V(N_f)$
- Ginzburg-Landau paradigm for second order (or weakly first order) transitions:
 - i.) there exists a local order parameter Φ
 - ii.) the UV free energy (Γ_Λ) can be expanded in terms of Φ
 - iii.) Γ_Λ has to reflect all symmetries

Renormalization group analysis of the chiral transition

- GL theory for the chiral transition:
 - gauge degrees of freedom are integrated out
 - the emerging order parameter (Φ) is a $N_f \times N_f$ matrix
 - chiral symmetry acts as $\Phi \rightarrow L\Phi R^\dagger$
- The most general UV free energy functional:

$$\begin{aligned}\Gamma_\Lambda = \int_x \left[& m^2 \text{Tr}(\Phi^\dagger \Phi) + g_1 (\text{Tr}(\Phi^\dagger \Phi))^2 + g_2 \text{Tr}(\Phi^\dagger \Phi \Phi^\dagger \Phi) + \dots \right. \\ & \left. + a(\det \Phi^\dagger + \det \Phi) + \text{Tr}(\partial_i \Phi^\dagger \partial_i \Phi) + \dots \right]\end{aligned}$$

- In essence identical with the previous theory!
 - BUT it is dimensionally reduced $\Rightarrow d = 3$
- Note: expansion of the full free energy is not allowed!
 - at T_C long wavelength fluctuations are important

Renormalization group analysis of the chiral transition

- Pisarski & Wilczek analysis of the Ginzburg–Landau theory ¹:
 - one-loop calculation of the β functions (no anomaly)
 - counterterms for g_1, g_2 :

$$\delta g_1, \delta g_2 \sim \text{Diagram: two external lines meeting at a vertex connected by a loop}$$

- Results (ϵ expansion, $\epsilon = 4 - d$):

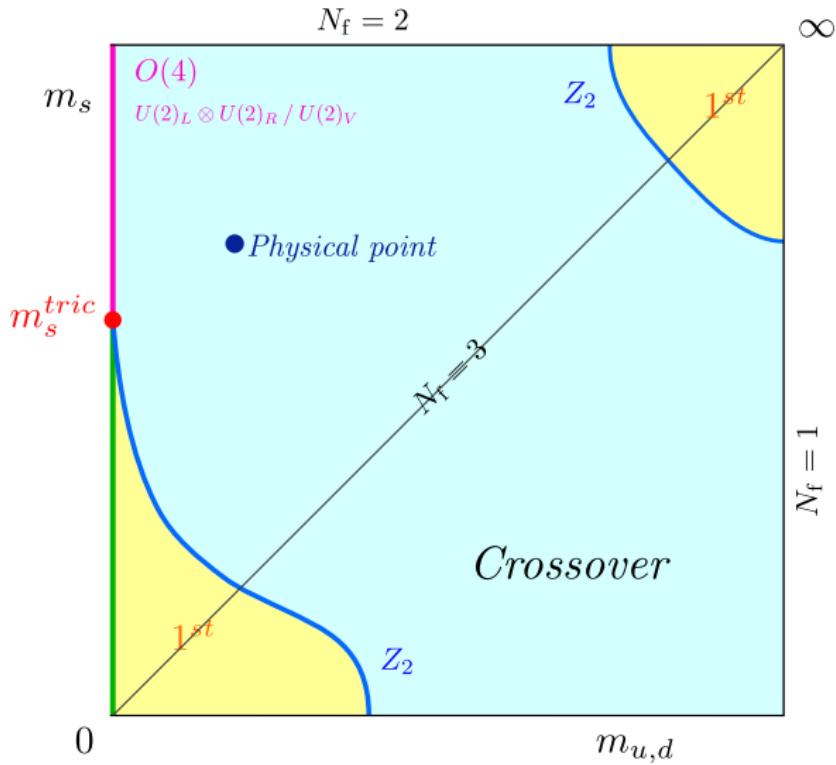
$$\begin{aligned}\beta_{g_1} &= -\epsilon g_1 + \frac{N_f^2 + 4}{4\pi^2} g_1^2 + \frac{N_f}{\pi^2} g_1 g_2 + \frac{3g_2^2}{4\pi^2} \\ \beta_{g_2} &= -\epsilon g_2 + \frac{3}{2\pi^2} g_1 g_2 + \frac{N_f}{2\pi^2} g_2^2\end{aligned}$$

- No infrared stable fixed point if $N_f > \sqrt{3}$
⇒ 2nd order transition cannot occur!
- Inclusion of the anomaly:
 - $N_f = 2$: second order transition with $O(4)$ exponents
 - $N_f = 3$: first order transition

¹R. D. Pisarski and F. Wilczek, Phys. Rev. D29, 338 (1984)

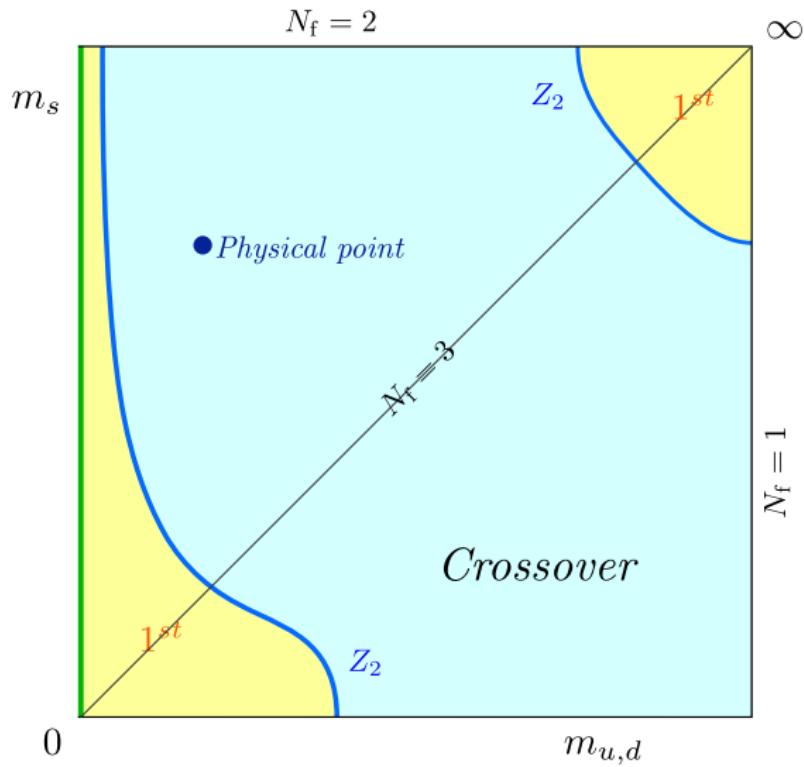
Renormalization group analysis of the chiral transition

Columbia plot with anomaly: [figure taken from F. Cuteri et. al, JHEP11, 141 (2021)]



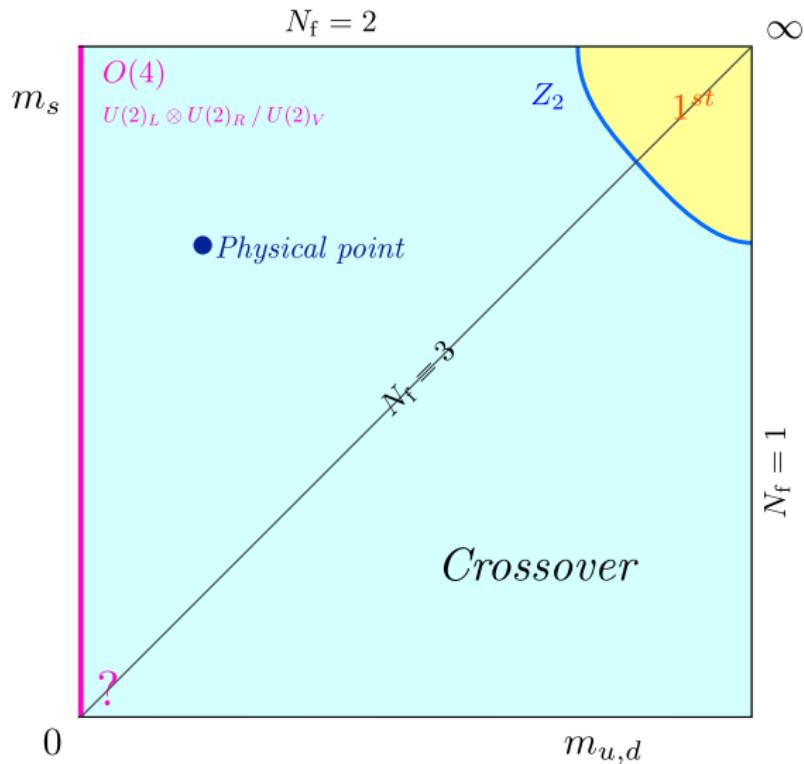
Renormalization group analysis of the chiral transition

Columbia plot without anomaly: [figure taken from F. Cuteri et. al, JHEP11, 141 (2021)]



Renormalization group analysis of the chiral transition

New, conjectured Columbia plot: [figure taken from F. Cuteri et. al, JHEP11, 141 (2021)]



Renormalization group analysis of the chiral transition

- Potential problems with the Pisarski & Wilczek analysis:
 - it uses the field theoretical RG \Rightarrow valid only close to the Gaussian fixed point
 - in $d = 3$ there are more (perturbatively) renormalizable operators!
 - ϵ expansion is not reliable
- Example: superconducting phase transition
 - Abelian Higgs model: ϵ expansion predicts a first order transition
 - Monte Carlo simulations showed that the transition can be of second order
 - IR fixed point is inaccessible in the ϵ expansion,
FRG is needed! ²

²GF & T. Hatsuda, Phys. Rev. D93, 121701 (2016).

GF & T. Hatsuda, Phys. Rev. D96, 056018 (2017)

Renormalization group analysis of the chiral transition

- How to build up the most general Ginzburg–Landau potential for three flavors in $d = 3$ in terms of renormalizable operators?
- **Independent** invariant tensors are needed ($N_f = 3!$):

$$\begin{aligned}\rho &= \text{Tr}(\Phi^\dagger \Phi) \\ \tau &= \text{Tr}(\Phi^\dagger \Phi - \rho/3)^2 \\ \rho_3 &= \text{Tr}(\Phi^\dagger \Phi - \rho/3)^3\end{aligned}$$

- $U_A(1)$ breaking terms:

$$\Delta = \det \Phi^\dagger + \det \Phi, \quad \tilde{\Delta} = \cancel{\det \Phi^\dagger} - \cancel{\det \Phi}$$

→ $\tilde{\Delta}^2$ could work but it is **not independent**

$$\rightarrow \tilde{\Delta}^2 = \tilde{\Delta}^2 + 4\rho^3/27 - 2\rho\tau/3 + 4\rho_3/3$$

$$\rightarrow \det \Phi^\dagger \cdot \det \Phi = (\Delta^2 - \tilde{\Delta}^2)/4$$

Renormalization group analysis of the chiral transition

- The most general Ginzburg–Landau potential (9 couplings!):

$$\Gamma_k[\Phi] = \int_x \left[\text{Tr} (\partial_i \Phi^\dagger \partial_i \Phi) + V_k[\Phi] \right]$$

$$V_k[\Phi] = m_k^2 \rho + a_k \Delta + g_{1,k} \rho^2 + g_{2,k} \tau \\ + b_k \rho \Delta + \lambda_{1,k} \rho^3 + \lambda_{2,k} \rho \tau + a_{2,k} \Delta^2 + g_{3,k} \rho_3 + \mathcal{O}(\Phi^7)$$

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$$V_k[\Phi] = m_k^2 \rho + a_k \Delta + g_{1,k} \rho^2 + g_{2,k} \tau$$

$$+ b_k \rho \Delta + \lambda_{1,k} \rho^3 + \lambda_{2,k} \rho \tau + a_{2,k} \Delta^2 + g_{3,k} \rho_3 + \mathcal{O}(\Phi^7)$$

- Flow equation:

$$\partial_k \Gamma_k = \frac{1}{2} \int_q \int_p \text{Tr} [\partial_k R_k(q, p) (\Gamma_k^{(2)} + R_k)^{-1}(-p, -q)]$$

- Left hand side? $\Rightarrow \beta$ functions!
- Right hand side? Need to compatible with the lhs!
 - $\rightarrow \Phi = \sum_{a=0}^8 \phi^a T^a \equiv \sum_{a=0}^8 (s^a + i\pi^a) T^a$
 - \rightarrow entries of $\Gamma_k^{(2)}$ depend on the fields, **not the invariants!**
 - \rightarrow trick: one is free to choose the background field

Renormalization group analysis of the chiral transition

$$\begin{aligned}
\beta_{m^2} &\equiv k \partial_k \bar{m}_k^2 = -2\bar{m}_k^2 - \frac{4}{9\pi^2} \frac{15\bar{g}_{1,k} + 4\bar{g}_{2,k}}{(1+\bar{m}_k^2)^2} + \frac{4}{3\pi^2} \frac{\bar{a}_k^2}{(1+\bar{m}_k^2)^3}, \\
\beta_a &\equiv k \partial_k \bar{a}_k = -\frac{3\bar{a}_k}{2} - \frac{4}{\pi^2} \frac{\bar{b}_k}{(1+\bar{m}_k^2)^2} + \frac{4}{3\pi^2} \frac{\bar{a}_k(3\bar{g}_{1,k} - 4\bar{g}_{2,k})}{(1+\bar{m}_k^2)^3}, \\
\beta_{g_1} &\equiv k \partial_k \bar{g}_{1,k} = -\bar{g}_{1,k} - \frac{1}{9\pi^2} \frac{2\bar{a}_{2,k} + 9\bar{\lambda}_{1,k} + 16\bar{\lambda}_{2,k}}{(1+\bar{m}_k^2)^2} + \frac{4}{27\pi^2} \frac{24\bar{a}_k \bar{b}_k + 117\bar{g}_{1,k}^2 + 48\bar{g}_{1,k}\bar{g}_{2,k} + 16\bar{g}_{2,k}^2}{(1+\bar{m}_k^2)^3} \\
&\quad - \frac{16}{9\pi^2} \frac{\bar{a}_k^2(\bar{g}_{1,k} + \bar{g}_{2,k})}{(1+\bar{m}_k^2)^4} + \frac{8}{9\pi^2} \frac{\bar{a}_k^4}{(1+\bar{m}_k^2)^5}, \\
\beta_{g_2} &\equiv k \partial_k \bar{g}_{2,k} = -\bar{g}_{2,k} + \frac{1}{3\pi^2} \frac{\bar{a}_{2,k} - 5\bar{g}_{3,k} - 13\bar{\lambda}_{2,k}}{(1+\bar{m}_k^2)^2} - \frac{4}{3\pi^2} \frac{\bar{a}_k \bar{b}_k - 6\bar{g}_{1,k}\bar{g}_{2,k} - 4\bar{g}_{2,k}^2}{(1+\bar{m}_k^2)^3} + \frac{4}{3\pi^2} \frac{\bar{a}_k^2(3\bar{g}_{1,k} + 5\bar{g}_{2,k})}{(1+\bar{m}_k^2)^4} \\
&\quad + \frac{2}{3\pi^2} \frac{\bar{a}_k^4}{(1+\bar{m}_k^2)^5}, \\
\beta_b &\equiv k \partial_k \bar{b}_k = -\frac{\bar{b}_k}{2} + \frac{4}{9\pi^2} \frac{\bar{b}_k(66\bar{g}_{1,k} - 4\bar{g}_{2,k}) + 3\bar{a}_k(5\bar{a}_{2,k} + 9\bar{\lambda}_{1,k} - 4\bar{\lambda}_{2,k})}{(1+\bar{m}_k^2)^3} \\
&\quad + \frac{8}{3\pi^2} \frac{-3\bar{a}_k^2 \bar{b}_k - 18\bar{a}_k \bar{g}_{1,k}^2 + 12\bar{a}_k \bar{g}_{1,k} \bar{g}_{2,k} + 4\bar{a}_k \bar{g}_{2,k}^2}{(1+\bar{m}_k^2)^4} + \frac{32}{9\pi^2} \frac{\bar{a}_k^3(3\bar{g}_{1,k} - \bar{g}_{2,k})}{(1+\bar{m}_k^2)^5}, \\
\beta_{\lambda_1} &\equiv k \partial_k \bar{\lambda}_{1,k} = \frac{8}{27\pi^2} \frac{9\bar{b}_k^2 + 3\bar{a}_{2,k}\bar{g}_{1,k} + 24\bar{g}_{1,k}(9\bar{\lambda}_{1,k} + \bar{\lambda}_{2,k}) + 4\bar{g}_{2,k}(9\bar{\lambda}_{1,k} + 4\bar{\lambda}_{2,k})}{(1+\bar{m}_k^2)^3} \\
&\quad - \frac{4}{81\pi^2} \frac{72\bar{a}_k \bar{b}_k(9\bar{g}_{1,k} + \bar{g}_{2,k}) + 4(207\bar{g}_{1,k}^2 + 108\bar{g}_{1,k}^2 \bar{g}_{2,k} + 72\bar{g}_{1,k} \bar{g}_{2,k}^2 + 16\bar{g}_{2,k}^3) + 9\bar{a}_k^2(2\bar{a}_{2,k} + 45\bar{\lambda}_{1,k} + 4\bar{\lambda}_{2,k})}{(1+\bar{m}_k^2)^4} \\
&\quad + \frac{32}{81\pi^2} \frac{\bar{a}_k^2(15\bar{a}_k \bar{b}_k + 171\bar{g}_{1,k}^2 + 36\bar{g}_{1,k} \bar{g}_{2,k} + 8\bar{g}_{2,k})}{(1+\bar{m}_k^2)^5} - \frac{80}{81\pi^2} \frac{\bar{a}_k^4(15\bar{g}_{1,k} + \bar{g}_{2,k})}{(1+\bar{m}_k^2)^6} + \frac{8}{9\pi^2} \frac{\bar{a}_k^6}{(1+\bar{m}_k^2)^7}, \\
\beta_{\lambda_2} &\equiv k \partial_k \bar{\lambda}_{2,k} = \frac{2}{9\pi^2} \frac{2\bar{g}_{2,k}(25\bar{g}_{3,k} + 54\bar{\lambda}_{1,k} + 44\bar{\lambda}_{2,k} - 2\bar{a}_{2,k}) - 9\bar{b}_k^2 - 6\bar{g}_{1,k}(\bar{a}_{2,k} - 5\bar{g}_{3,k} - 28\bar{\lambda}_{2,k})}{(1+\bar{m}_k^2)^3} \\
&\quad + \frac{1}{3\pi^2} \frac{36\bar{a}_k \bar{b}_k(2\bar{g}_{1,k} + \bar{g}_{2,k}) - 8\bar{g}_{2,k}(36\bar{g}_{1,k}^2 + 21\bar{g}_{1,k} \bar{g}_{2,k} + 7\bar{g}_{2,k}^2) + \bar{a}_k^2(6\bar{a}_{2,k} + 5\bar{g}_{3,k} + 36\bar{\lambda}_{1,k})}{(1+\bar{m}_k^2)^4} \\
&\quad + \frac{8}{27\pi^2} \frac{9\bar{a}_k^3 \bar{b}_k + 180\bar{a}_k^2 \bar{g}_{1,k}^2 + 132\bar{a}_k^2 \bar{g}_{1,k} \bar{g}_{2,k} + 26\bar{a}_k^2 \bar{g}_{2,k}^2}{(1+\bar{m}_k^2)^5} + \frac{20}{9\pi^2} \frac{\bar{a}_k^4(3\bar{g}_{1,k} + 2\bar{g}_{2,k})}{(1+\bar{m}_k^2)^6}, \\
\beta_{a_2} &\equiv k \partial_k \bar{a}_{2,k} = \frac{4}{3\pi^2} \frac{6\bar{b}_k^2 + 15\bar{a}_{2,k}\bar{g}_{1,k} - 8\bar{a}_{2,k}\bar{g}_{2,k}}{(1+\bar{m}_k^2)^3} + \frac{16}{\pi^2} \frac{\bar{a}_k \bar{b}_k(\bar{g}_{2,k} - 3\bar{g}_{1,k})}{(1+\bar{m}_k^2)^4} + \frac{16}{3\pi^2} \frac{\bar{a}_k^2(9\bar{g}_{1,k}^2 + 2\bar{g}_{2,k})}{(1+\bar{m}_k^2)^5}, \\
\beta_{g_3} &\equiv k \partial_k \bar{g}_{3,k} = \frac{4}{3\pi^2} \frac{15\bar{g}_{1,k}\bar{g}_{3,k} + \bar{g}_{2,k}(2\bar{a}_{2,k} + \bar{g}_{3,k} + 12\bar{\lambda}_{2,k})}{(1+\bar{m}_k^2)^3} \\
&\quad + \frac{1}{\pi^2} \frac{4\bar{a}_k \bar{b}_k \bar{g}_{2,k} + 8\bar{g}_{2,k}^2(\bar{g}_{2,k} - 9\bar{g}_{1,k}) + \bar{a}_k^2(\bar{g}_{3,k} + 8\bar{\lambda}_{2,k} - 2\bar{a}_{2,k})}{(1+\bar{m}_k^2)^4} + \frac{16}{9\pi^2} \frac{3\bar{a}_k^3 \bar{b}_k + 2\bar{a}_k^2 \bar{g}_{2,k}(7\bar{g}_{2,k} - 12\bar{g}_{1,k})}{(1+\bar{m}_k^2)^5} \\
&\quad + \frac{20}{9\pi^2} \frac{\bar{a}_k^4(5\bar{g}_{2,k} - 6\bar{g}_{1,k})}{(1+\bar{m}_k^2)^6} + \frac{2}{\pi^2} \frac{\bar{a}_k^6}{(1+\bar{m}_k^2)^7}.
\end{aligned}$$

Renormalization group analysis of the chiral transition

- Fixed points: $\beta_i = 0 \forall i$
- First step: solve for the marginal couplings
→ $\beta_{\lambda_1} = \beta_{\lambda_2} = \beta_{a_2} = \beta_{g_3} \equiv 0$
→ $\lambda_1, \lambda_2, a_2, g_3$ are plugged into the remaining β functions
- Second step: solve for the relevant couplings
- Third step: check stability matrix $\partial\beta_i/\partial\omega_j$
($\{\omega_j\}$: m^2, g_1, g_2, a, b)

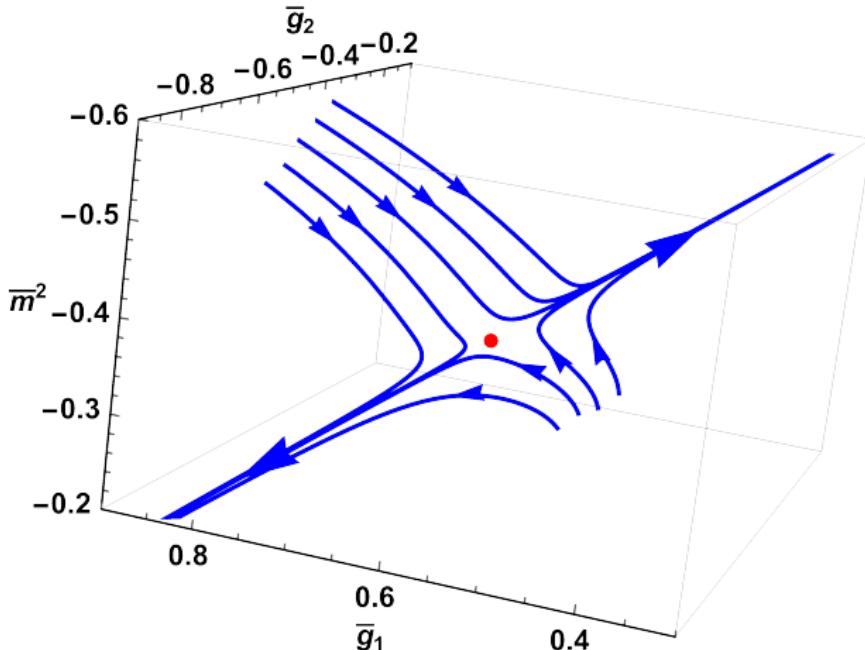
m^2	g_1	g_2	a	b	# of RD
0	0	0	0	0	5
-0.31496	0.43763	0	0	0	3
-0.38262	0.59726	-0.62042	0	0	2
-0.01786	0.09163	-0.14148	-0.11900	0.39087	4

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- No fixed point with **one relevant direction**
→ first order transition?
- BUT the third one has a **block diagonal stability** matrix:
→ $(m^2, g_1, g_2) \oplus (a, b)$
- Both a and b are related to the $U_A(1)$ anomaly!
→ without anomaly the **no. of relevant directions is 1 !**

Renormalization group analysis of the chiral transition



- If the $U_A(1)$ symmetry is recovered at T_c , the transition is of second order!
- Temperature eigenvalue leads to $\nu \approx 0.83$

Renormalization group analysis of the chiral transition

- Lessons from the ϵ -expansion:
 - if the $U_A(1)$ symmetry is recovered at T_c :
first order transition for $N_f = 2, 3$
 - if the anomaly is present at T_c :
second order for $N_f = 2$, first order for $N_f = 3$

Renormalization group analysis of the chiral transition

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- Lessons from the FRG directly in $d = 3$ for $N_f = 3$:
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Renormalization group analysis of the chiral transition

- Lessons from the ϵ -expansion:
 - if the $U_A(1)$ symmetry is recovered at T_c :
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- Lessons from the FRG directly in $d = 3$ for $N_f = 3$:
 - if the $U_A(1)$ symmetry is recovered at T_c :
second order
 - if the anomaly is present at T_c :
first order
- Increasing evidence of a second order transition for $N_f = 3$:
 - F. Cuteri, O. Philipsen, A. Sciarra, JHEP **11**, 141 (2021)
 - L. Dini et al., Phys. Rev. D**105**, 034510 (2022)
- If the transition is of second order, RG hints that the **$U_A(1)$ axial symmetry is recovered at T_c !**

Summary

- Thermal evolution of the $U_A(1)$ anomaly in L σ M ($N_f = 3$)
 - **resummation**: infinite class of $U_A(1)$ breaking operators
 - resulting anomaly function calculated via the **FRG**
 - mesonic fluctuations **strengthen the anomaly** toward T_C
 - **instanton effects** introduced via bare anomaly parameter
 - ~ 10% increase of the anomaly coupling below T_C

Summary

- Thermal evolution of the $U_A(1)$ anomaly in L σ M ($N_f = 3$)
 - **resummation**: infinite class of $U_A(1)$ breaking operators
 - resulting anomaly function calculated via the **FRG**
 - mesonic fluctuations **strengthen the anomaly** toward T_C
 - **instanton effects** introduced via bare anomaly parameter
 - ~ 10% increase of the anomaly coupling below T_C
- Analysis of chiral phase transition in the chiral limit ($N_f = 3$)
 - recalculating RG flows in the GL theory (**9 couplings!**)
 - **2 new fixed points** in the system were found
 - without the anomaly, one of them describes a **second order thermal phase transition**
 - if the transition is of second order, the bare anomaly parameter may need to be **set to zero at T_c**

Summary

- Questions to be asked:

- higher order $U_A(1)$ breaking operators?
($|Q| > 1$ instanton charges)
- what is the T dependence of the bare anomaly coupling?
- improving the truncation in the GL potential?
- transition order for general N_f ?