Aspects of topology and confinement in large-N gauge theories

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Talk based on:

CB, Bonati, Papace, Vadacchino, *The θ-dependence of the Yang–Mills spectrum from analytic continuation*, JHEP **05** (2024) 163 [2402.03096]

CB, D'Elia, Verzichelli, *The θ-dependence of the* SU(N) *critical temperature at large* N, JHEP **02** (2024) 156 [2312.12202]

CB, *The topological susceptibility slope χ' of the pure-gauge* SU(3) *Yang–Mills theory*, JHEP **01** (2024) 116 [2311.06646]

CB, D'Elia, Lucini, Vadacchino, *Towards glueball masses of large-N* SU(N) *pure-gauge theories without topological freezing*, Phys. Lett. B **833** (2022) 137281 [2205.06190]

CB, Bonati, D'Elia, *Large-N* SU(N) *Yang–Mills theories with milder topological freezing*, JHEP **03** (2021) 111 [2012.14000]

The topological term in QCD

Renormalizability, Lorentz-invariance and gauge-invariance allow for a gluonic θ-term in the QCD action:

$$S_{\mathrm{QCD}}(\theta) = \frac{1}{2}\int d^4x \, \mathrm{Tr}\left[G_{\mu\nu}(x)G^{\mu\nu}(x)\right] + \int d^4x \sum_{f=1}^{N_f} \overline{\psi}(\not{D} + m_f)\psi + \theta Q$$

$$Q = \frac{g^2}{32\pi^2}\int d^4x \, \varepsilon_{\mu\nu\rho\sigma} \, \mathrm{Tr}\left[G^{\mu\nu}(x)G^{\rho\sigma}(x)\right] \in \mathbb{Z}$$

The topological charge Q is a gauge-invariant integer quantity corresponding to the number of windings of the gauge field $A_\mu(x)$ around the group manifold at $x \to \infty$.

Such coupling introduces a **non-trivial dependence on** θ:

$$\mathcal{Z}_{\mathrm{QCD}}(\theta) = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{iS_{\mathrm{QCD}} + i\theta Q} = \sum_{n=-\infty}^{\infty} e^{i\theta n}\int_{Q=n} \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{iS_{\mathrm{QCD}}}.$$

Physical relevance of θ-dep. in QCD and related theories

The relevance of θ-dependence relevance is broad,
covering both theoretical and phenomenological aspects of the Standard Model.

Phenomenology

- Non-zero $\theta \to$ Explicit breaking of CP symmetry
  $\implies$ non-zero neutron Electric Dipole Moment (nEDM).
  Experiments found nEDM well compatible with zero $\implies |\theta^{(\mathrm{exp})}| \lesssim 10^{-9} - 10^{-10}$
- No CP violation from QCD $\implies$ fine-tuning problem on θ: **strong-**CP **problem**
  $\implies$ new physics beyond Standard Model to explain it (Peccei–Quinn axion)

Theory

- Q **breaks** the $\mathrm{U}(1)_{\mathrm{A}}$ flavor symmetry through **anomaly** $\implies$ large mass of η' meson. Physics of the η' related to QCD θ-dependence in the ideal limit of large number of colors: $N_c \to \infty$
- A non-vanishing θ changes the QCD vacuum $\implies$ Does θ change the confining properties of strong interactions? How and why?
- **θ-dep. in $2d$ theories**: e.g., $2d$ CP^{N-1} models, $2d$ $\mathrm{U}(N)$ Yang–Mills, ...
  Extensively studied in the large-N limit (analogy with large N_c)

Topology and the axial anomaly

At energy scales $\ll \Lambda$ but $\gg m_u, m_d \sim m_l$, lightest quark masses are small perturbation of **massless QCD**. Massless QCD enjoys a global flavor chiral symmetry:

$$\mathrm{G} = \mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R} \otimes \mathrm{U(1)_V} \otimes \mathrm{U(1)_A}$$

The $\mathrm{U(1)_A}$ classical invariance is *anomalous* (functional measure not invariant) $\implies$ not a symmetry of the quantum theory! (t 'Hooft, PRL 37, 1976)
Anomaly proportional to the gluon topological charge:

$$\begin{array}{c} \psi_\mathrm{L} \to e^{i\alpha}\psi_\mathrm{L} \\ \psi_\mathrm{R} \to e^{-i\alpha}\psi_\mathrm{R} \end{array} \quad \implies \quad \mathcal{D}\overline{\psi}\mathcal{D}\psi \to \mathcal{D}\overline{\psi}\mathcal{D}\psi\, e^{i2N_f\alpha Q}$$

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}A e^{iS_{\mathrm{QCD}}^{(m_f=0)}+i\theta Q} \underset{\mathrm{U(1)_A}}{\longrightarrow} \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}A e^{iS_{\mathrm{QCD}}^{(m_f=0)}+i\theta Q} e^{i2N_f\alpha Q}$$

$$\implies S_{\mathrm{QCD}}^{(m_f=0)}(\theta) \underset{\mathrm{U(1)_A}}{\longrightarrow} S_{\mathrm{QCD}}^{(m_f=0)}(\theta + 2N_f\alpha)$$

This means that, in the chiral limit, QCD is **θ-independent** because of the axial anomaly, as the θ-term is reabsorbed via quark field redefinition: $\alpha = -\theta/(2N_f)$.

This apparently seems in contradiction with the expected behavior of QCD when the number of colors $N \to \infty$.

The large-N limit of QCD

Let us consider the 't Hooft limit $N \to \infty$ (N_f fixed) with $g^2 N = \lambda$ constant:

$$\frac{g^2_{1\mathrm{loop}}(\mu)}{4\pi^2} = \frac{3}{(11N - 2N_f)\log(\mu/\Lambda)} \implies g^2 \underset{N\to\infty}{\sim} \mathcal{O}\left(\frac{1}{N}\right) \text{ if } \Lambda \underset{N\to\infty}{\sim} \mathcal{O}(N^0).$$

At large-N the loop-diagrammatic expansion of QCD can be systematically rearranged in powers of $1/N$ (*non-perturbative* in λ) (t 'Hooft, NPB 72, 1974)

- Leading order: Purely-gluonic planar diagrams
- Sub-leading order: planar diagrams with one quark line, suppressed as $1/N$.

Apparent contradiction $\longrightarrow$ U(1)$_\mathrm{A}$ puzzle

- $m_l \to 0$ at fixed N:
  QCD becomes θ-independent in the chiral limit because of the anomaly for any fixed value of N. Cancellation due to dynamical quark contribution.
- $N \to \infty$ at fixed m_l:
  quark loops are suppressed compared to the gluonic contribution $\implies$ quarks decouple, QCD becomes **pure-gauge theory** in the large-N limit

How can anomaly cancel θ-dependence at large-N in the chiral limit?

Phenomenological role of the topological susceptibility

Key quantity to solve the $U(1)_A$ puzzle: the *topological susceptibility* χ

$$\chi \equiv \lim_{V\to\infty} \frac{\langle Q^2 \rangle}{V}$$

$$= \frac{1}{V}\int d^4x\, d^4y\ \langle q(x)q(y)\rangle = \int d^4x\ \langle q(x)q(0)\rangle$$

$$Q = \frac{g^2}{32\pi^2}\int d^4x\, \varepsilon_{\mu\nu\rho\sigma}\, \mathrm{Tr}\left[G^{\mu\nu}(x)G^{\rho\sigma}(x)\right] \equiv \int d^4x\, q(x)$$

The topological susceptibility can be seen as the leading "response" of the QCD partition function when a θ-term is turned on:

$$\frac{\mathcal{Z}_{\mathrm{QCD}}(\theta)}{\mathcal{Z}_{\mathrm{QCD}}(0)} = \langle e^{i\theta Q}\rangle_{\theta=0} = 1 + \frac{\chi\theta^2}{2} + \dots$$

$$\Longrightarrow\ \chi = 0 \text{ in the chiral limit!}$$

$$\chi \propto \frac{d^2\mathcal{Z}_{\mathrm{QCD}}(\theta)}{d\theta^2} = 0 \text{ because } \mathcal{Z}_{\mathrm{QCD}} \text{ is } \theta\text{-independent when } m_l \to 0$$

Witten–Veneziano solution to $U(1)_A$ puzzle

Large-N expansion of topological charge density correlator:

$$G_{QCD}(p^2) \equiv \int d^4x e^{ip\cdot x} \langle q(x)q(0)\rangle$$

$$G_{QCD}(p^2) = G_{YM}(p^2) - \sum_{\text{mesons}} \frac{|A_n|^2}{p^2+m_n^2} + \mathcal{O}\left(1/N^2\right)$$

$$p^2 = 0 \quad \Longrightarrow \quad \chi_{QCD} = \chi_{YM} - \sum_{\text{mesons}} \frac{|A_n|^2}{m_n^2}$$

$\mathcal{O}(N^0)$: pure-Yang–Mills contribution χ_{YM}

$\mathcal{O}(N^{-1})$: contribution of mesonic state propagation: $m_n^2 \sim N^0$, $|A_n|^2 \sim N^{-1}$

Solution: there is a meson state (η') with $m^2 \sim \mathcal{O}(1/N)$ which cancels pure-gauge contribution to χ_{QCD} (Witten, NPB 149, 1979; Veneziano, NPB 156, 1979)

$$0 \underset{m\to 0}{=} \chi_{QCD} = \chi_{YM} - |A_{\eta'}|^2/m_{\eta'}^2 \quad \Longrightarrow \quad \chi_{YM} = |A_{\eta'}|^2/m_{\eta'}^2 \sim \mathcal{O}(N^0).$$

$$|A_{\eta'}|^2 = F_\pi^2 m_{\eta'}^4/6 \implies \chi_{YM} = F_\pi^2 m_{\eta'}^2/6$$

η' massless Nambu–Goldstone boson not in the chiral but in the large-N limit!

Large-N θ-dependence of Yang–Mills vacuum energy

Witten–Veneziano + general arguments constrain θ-dependence of vacuum energy:

$$E(\theta) = -\frac{1}{V}\log\left[\frac{Z_{\rm YM}(\theta)}{Z_{\rm YM}(0)}\right] \qquad E(\theta) = \frac{1}{2}\chi\theta^2\left(1 + b_2\theta^2 + b_4\theta^4 + \dots\right)$$

$$\chi = \left.\frac{\langle Q^2\rangle}{V}\right|_{\theta=0} \qquad b_2 = -\frac{1}{12}\left.\frac{\langle Q^4\rangle - 3\langle Q^2\rangle^2}{\langle Q^2\rangle}\right|_{\theta=0} \qquad b_{2n} \propto \langle Q^{2n}\rangle_c$$

- $\theta Q \propto \theta g^2 \varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}(G^{\mu\nu}G^{\rho\sigma}) \sim \mathcal{O}(\lambda\,\theta/N) \implies$ actual expansion parameter: θ/N
- $E \sim \mathcal{O}(N^2)$ (number of color degrees of freedom)

$$\implies E(\theta, N) \underset{N\to\infty}{\sim} N^2 f\left(\frac{\theta}{N}\right)$$ (Witten, PRL 81, 1998)

Witten–Veneziano requires: $\chi_{\rm YM}(N) = \overline{\chi}_{\rm YM} + \mathcal{O}(1/N^2)$, $\qquad \overline{\chi}_{\rm YM} \sim \mathcal{O}(N^0)$

$$\implies b_2(N) = \frac{\bar{b}_2}{N^2}[1 + \mathcal{O}(1/N^2)] \qquad \left(b_{2n}(N) \underset{N\to\infty}{\sim} 1/N^{2n}\right)$$

How to check this scenario? θ-dependence intrinsically non-perturbative

$\implies$ Needs non-perturbative first-principle methods: lattice numerical approach

What are the main difficulties found using this approach?

Topology freezing

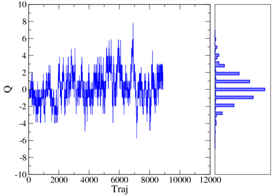
$\mathcal{P} \propto e^{-S_{\mathrm{E}}}$ sampled with Monte Carlo methods to obtain representative ensemble $\{\mathcal{O}_i\} \to$

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}\phi \, e^{-S_{\mathrm{E}}[\phi]} \mathcal{O}[\phi]}{\int \mathcal{D}\phi \, e^{-S_{\mathrm{E}}[\phi]}}$$ obtained from ensemble mean.

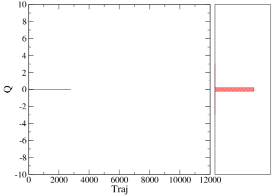
Approaching the continuum limit, customary algorithms experience **Topological Freezing** $\to$ machine time needed to generate representative ensemble of Q grows **exponentially**.

Severity increases at large N.

$N = 3$, $a \simeq 0.056$ fm$\simeq (3.5\ \mathrm{GeV})^{-1}$



$N = 3$, $a \simeq 0.039$ fm $\simeq (5.1\ \mathrm{GeV})^{-1}$

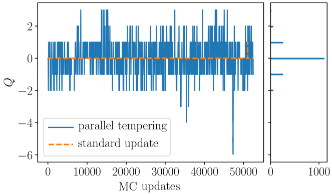


Parallel Tempering on Boundary Conditions

Proposed in (Hasenbusch, 2017 [1706.04443]) for $2d$ CP^{N-1}.

First applied to $4d$ $\mathrm{SU}(N)$ in (CB et al., 2021 [2012.14000]).

$N = 6$, $a \simeq 0.089$ fm $\simeq (2.2\ \mathrm{GeV})^{-1}$



Combine simulations with **periodic** and **open** boundaries.

Open: Q fluctuates a lot because is no more constrained to be integer, but boundaries introduce systematics.

Periodic: Q is measured with periodic boundaries, where unphysical effects due to the open boundaries are avoided.

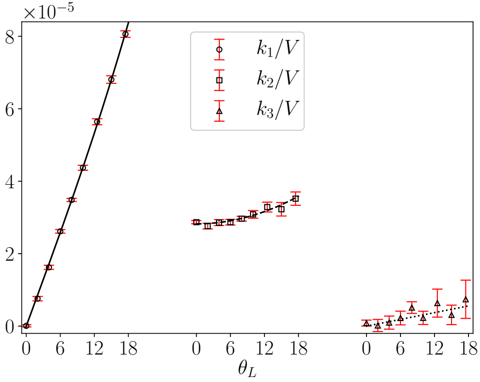
Simulate simultaneously systems with varying boundary conditions with swaps of configurations and take best of both worlds!

Higher-order cumulants and imaginary-θ simulations

$$S_{\mathrm{YM}}^{(\mathrm{E})} = S_{\theta=0} + i\theta Q \implies \text{at non-zero } \theta \text{ the Euclidean action is complex}$$
$$\implies \text{sign problem, no Monte Carlo simulations}$$

Imaginary-θ simulations: sign problem avoided through analytic continuation, $\theta = i\theta_I$.

$$S \to S + \theta_I Q, \quad \theta_I \equiv i\theta \qquad \implies \qquad k_n \to k_n(\theta_I) = \langle Q^n \rangle_c (\theta_I) \propto \frac{d^n E_{\mathrm{YM}}(\theta_I)}{d\theta_I^n}$$

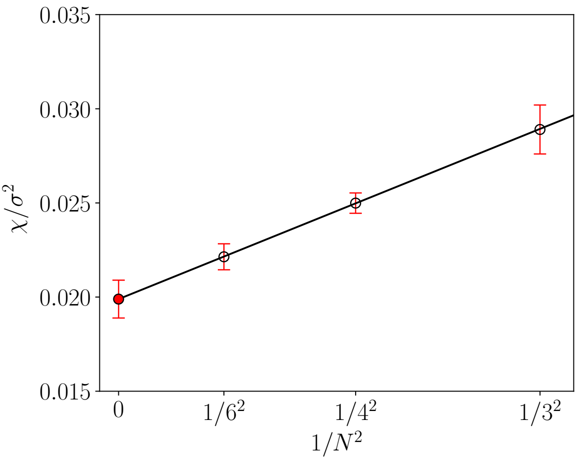


Further benefits:

- coupling Q to the action acts as a source term and improves signal-to-noise ratio.
- **Imaginary-θ fit** $\implies$ extract χ and b_2 from combined fit of θ_I-dependence:

$$\frac{\langle Q \rangle}{V}(\theta_I) = \chi(\theta_I - 2b_2\theta_I^3 + \dots)$$
$$\frac{\langle Q^2 \rangle - \langle Q \rangle^2}{V}(\theta_I) = \chi(1 - 6b_2\theta_I^2 + \dots)$$
$$\frac{\langle Q^3 \rangle_c}{V}(\theta_I) = \chi(-12b_2\theta_I + \dots)$$

Large-N limit of χ from the lattice



Parallel Tempering + Imaginary-θ

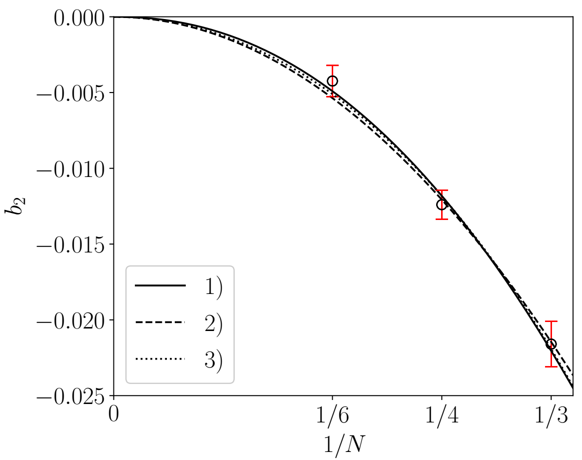
CB, Bonati, D'Elia, JHEP **03** (2021) 111 [arXiv:2012.14000]

Witten–Veneziano: $\chi_{\mathrm{YM}}^{1/4} \simeq 180\ \mathrm{MeV} + \mathcal{O}(1/N^2)$

Lattice: $\chi_{\mathrm{YM}}/\sigma^2 = 0.0199(10) + 0.08(2)/N^2$

$$\lim_{N\to\infty} \chi_{\mathrm{YM}}/\sigma^2 = 0.0199(10) \implies \lim_{N\to\infty} \chi_{\mathrm{YM}}^{1/4} = 173(8)\ \mathrm{MeV}$$

Large-N limit of b_2 from the lattice



Parallel Tempering + Imaginary-θ

CB, Bonati, D'Elia, JHEP **03** (2021) 111 [arXiv:2012.14000]

Large-N prediction: $b_2 = \bar{b}_2/N^2 + \mathcal{O}(1/N^4)$

1) $b_2 = \bar{b}_2/N^\gamma \longrightarrow \gamma = 2.17(26)$

2) $b_2 = \bar{b}_2/N^2 \longrightarrow \bar{b}_2 = -0.193(10)$

3) $b_2 = \bar{b}_2/N^2 + \bar{b}_2^{(1)}/N^4 \longrightarrow \bar{b}_2^{(1)} = -0.17(35)$

Witten–Veneziano mechanism beyond $p^2 = 0$

Witten–Veneziano mechanism assumes topological charge correlator to be dominated by the $p^2 = 0$ behavior (topological susceptibility χ)

$$G(p^2) = \int d^4x e^{ip\cdot x} \langle q(x)q(0)\rangle = \chi - p^2\chi' + \mathcal{O}(p^4), \quad \chi' = \frac{1}{8}\int d^4x\, |x|^2 \langle q(x)q(0)\rangle$$

$G(p^2 = 0) \sim G(p^2 \simeq m^2_{\eta'})$ for the Witten–Veneziano mechanism to hold $\implies |\chi'| \ll \chi/m^2_{\eta'}$ bound on the topological susceptibility slope χ'.

Using the $1/N$ expansion of $G(p^2)$ at large-N (CB, JHEP 01 (2024) 116 [2311.06646]):

$$\frac{\chi'_{\rm YM}}{N} = \left[\lim_{m\to 0} \frac{\chi'_{\rm QCD}}{N}\right] + \frac{F_\pi^2}{N} \simeq (12\ \mathrm{MeV})^2$$

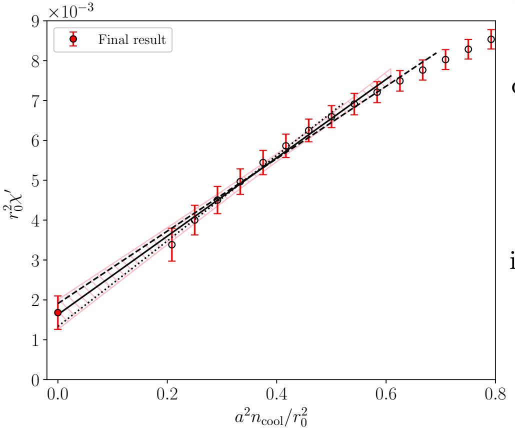
$$\implies \chi'/N \sim \mathcal{O}(N^0) \text{ at large } N$$

The numerical value of this prediction was obtained using:

$F_\pi^2/N \simeq [55(5)\ \mathrm{MeV}]^2$ (lattice large-N result, García Pérez et al., 2020 [2011.13061])

$\lim_{m\to 0} \chi'_{\rm QCD} \simeq -[32.8(2.4)\ \mathrm{MeV}]^2$ (Chir. Pert. Theo., Leutwyler, 2000 [hep-ph/0008124])

First reliable lattice determination of χ' in SU(3)



Main problem: correlators computed after *smoothing* to reduce noise. Sources are smeared up to radius r_s: $\langle q_{\rm smear}(x) q_{\rm smear}(0)\rangle$.

Correlator unphysical for $r < r_s$ $\implies$ using smeared correlators introduces a dependence of χ' on r_s.

Strategy: compute continuum limit at fixed r_s, then take $r_s \to 0$

(CB, JHEP 01 (2024) 116 [2311.06646])

$$\chi'(N=3) = [17.2(2.1)\ \mathrm{MeV}]^2$$

From same simulations: $\chi(N=3) = [200.4(3.6)\ \mathrm{MeV}]^4$

$\implies \chi'/(\chi/m_{\eta'}^2) \simeq 0.16 \implies$ **supports Witten–Veneziano mechanism**

$$\chi'/N|_{N=3} = [10.0(1.2)\ \mathrm{MeV}]^2$$

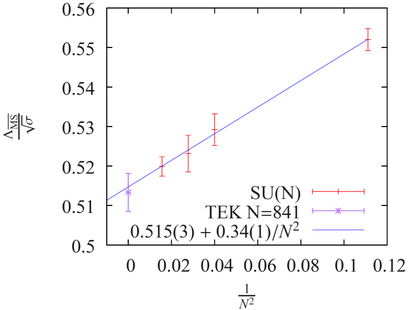
remarkably close to our large-N estimate $\lim_{N\to\infty} \chi'/N \simeq (12\ \mathrm{MeV})^2$.

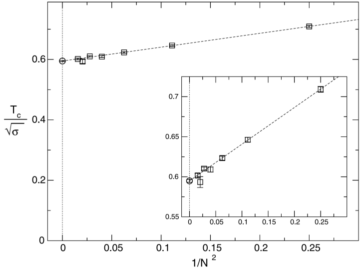
Confinement in the large-N limit

The 't Hooft large-N limit assumes $g^2 \sim \mathcal{O}(1/N)$, meaning that $\Lambda \sim \mathcal{O}(N^0)$. This assumes confinement to survive the large-N limit.

At $\theta = 0$ **well-verified assumption** from the lattice.

- **Dynamically-generated scale**: $\Lambda_{\overline{\mathrm{MS}}}(N=\infty)/\sqrt{\sigma} = 0.515(3)$
  (González-Arroyo & Okawa, 2013 [1206.0049])
- **Critical deconfinement temperature**: $T_c(N=\infty)/\sqrt{\sigma} = 0.595(2)$
  (Lucini et al., 2012 [1202.6684])





Relevant issue: what happens to confinement at non-vanishing θ?

The deconfinement transition at non-vanishing θ

Vacuum Energy ($T = 0$) $\longrightarrow$ Free Energy (finite-T)

$$f(T,\theta) = f(T,0) + \tfrac{1}{2}\chi(T)\theta^2 + \dots$$

$$f(T,0) = f(T_c,0) + \tfrac{T-T_c}{T_c}\epsilon + \dots$$

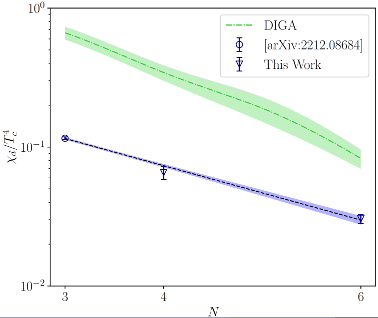
- Deconfinement transition is first order ($N > 2$).
- If still first order at non-zero θ, at $T_c(\theta)$ equal free energies in the two phases. Imposing $f_c(T_c(\theta),\theta) = f_d(T_c(\theta),\theta)$ we have: (D'Elia et al., 2012 [1205.0538])

$$T_c(\theta) = T_c(0)[1 - R\theta^2 + \mathcal{O}(\theta^4)], \quad R = \frac{1}{2}\frac{\Delta\chi(\theta=0)}{L(\theta=0)}.$$

- R related to properties of the $\theta = 0$ transition: $\Delta\chi = \chi_c - \chi_d, \quad L = \epsilon_d - \epsilon_c$ (Latent Heat)
  - $L \sim \mathcal{O}(N^2)$ (number of degrees of freedom)
  - $\chi_c \sim \mathcal{O}(N^0)$ (Witten–Veneziano)
  - $\chi_d \sim e^{-N}$ (CB et al., 2024 [2312.12202])

$\Longrightarrow \; R > 0 \sim \mathcal{O}(1/N^2)$

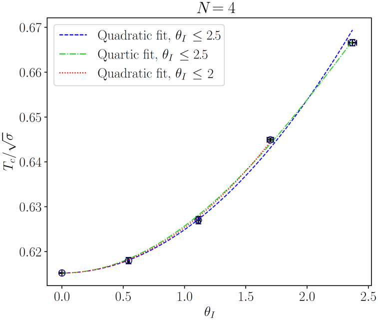
$\Longrightarrow \; T_c$ slightly reduced at non-zero θ

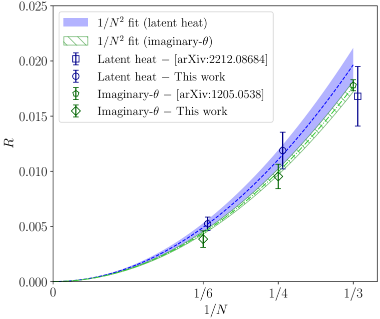


Large-N θ-dependence of T_c from the lattice

$N = 3$: deconf. transition exists at small-θ and is first-order (D'Elia et al., 2013 [1306.2919]), and imaginary-θ shows reduction of T_c (D'Elia et al., 2012 [1205.0538]). Predicted relation $R = \Delta\chi/(2L)$ holds for $N = 3$ (Borsanyi et al., 2023 [2212.08684]).

$N > 3$: using parallel tempering + imaginary-θ, we can show that this scenario remains true also in the large-N limit (CB et al., 2024 [2312.12202])





• From imaginary-θ fit

$R = \overline{R}/N^{\gamma} \longrightarrow \gamma = 2.20(24)$

$R = \overline{R}/N^{2} \longrightarrow \overline{R} = 0.159(4)$

$R = \frac{\overline{R}}{N^2} + \frac{\overline{R}^{(1)}}{N^4} \rightarrow \overline{R}^{(1)} = 0.22(25)$

• From latent heat perfectly agreeing results: $\overline{R} = 0.177(14)$.

The lattice thus fully confirms:

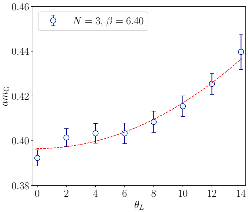
$R = \Delta\chi/(2L) = \overline{R}/N^2 + \mathcal{O}(1/N^4)$

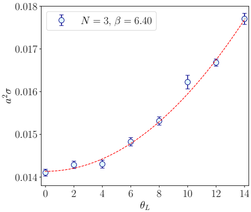
String tension and mass of lightest glueball state G ($\mathrm{G}(\theta = 0) = 0^{++}$) are found to decrease with θ (parallel tempering + imaginary-θ) (CB et al., 2024 [2402.03096])

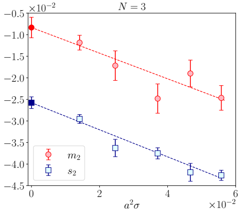
$\Longrightarrow$ perfectly fits with picture of reduction of confinement scales at non-zero θ

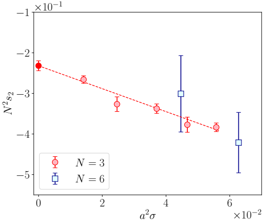
$$m_{\mathrm{G}}(\theta) = m_{0^{++}}(1 + m_2\theta^2 + \dots), \qquad \sigma(\theta) = \sigma(0)(1 + s_2\theta^2 + \dots)$$

$$m_2 = \overline{m}_2/N^2, \;\; s_2 = \overline{s}_2/N^2. \qquad \text{Lattice result: } \overline{m}_2 < 0, \;\; \overline{s}_2 < 0$$









θ-dep. not cancels in dimensionless ratios:

$$\frac{T_c(\theta)}{\sqrt{\sigma(\theta)}} \simeq 0.595(2) - 0.044(2)\left(\frac{\theta}{N}\right)^2 + \dots$$

$$\frac{m_{\mathrm{G}}(\theta)}{\sqrt{\sigma(\theta)}} \simeq 3.07(2) - 0.041(21)\left(\frac{\theta}{N}\right)^2 + \dots$$

Conclusions and take-home messages

- Combining Parallel Tempering + imaginary-θ it is possible to accurate study θ-dependence at large-N on the lattice beyond leading order
- At large-N and at non-vanishing (small) θ the lattice confirms effective dependence on θ/N, and expected large-N scaling already holds for $N \geq 3$
- At large-N the confined phase survives: all confining scales have finite large-N limit. With non-vanishing (small) θ confinement is "reduced" but not lost
- Above scenario confirmed by decrease of T_c, σ and $m_{0^{++}}$ at non-zero θ. Leading $\mathcal{O}(\theta^2)$ corrections are $\mathcal{O}(1/N^2)$ for these quantities
- Lattice results for χ and χ' support the large-N Witten–Veneziano mechanism, and show that $N = 3$ is remarkably "close" to $N = \infty$

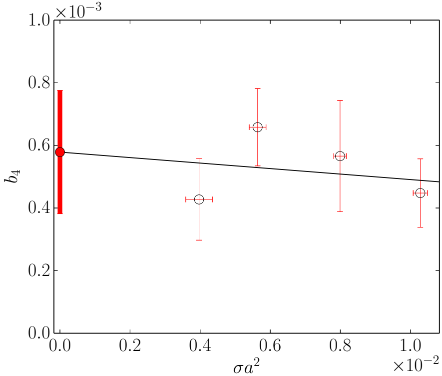
Some future outlooks

- $N = 3$ result for χ' very close to large-N prediction. Lattice large-N investigation?
- Recently, we showed that large-N limit of chiral condensate is very close to $N = 3$ value using large-N twisted volume-reduced models [1-site lattice but $N \sim \mathcal{O}(100)$] (CB et al., 2023 [2309.15540]).

  Chiral-symmetry restoration temperature at large N? How does it compare with deconfinement one?

Back-up Slides

Continuum limit of b_4 in SU(2) Yang–Mills theory



$$E_{\text{YM}}(\theta) - E_{\text{YM}}(0) = \frac{1}{2}\chi\theta^2(1 + b_2\theta^2 + b_4\theta^4 + \dots)$$

$N = 2$: $b_4 = 6(2) \cdot 10^{-4}$ (CB et al., 2019 [1807.11357])

$N = 3$: $|b_4| \lesssim 4 \cdot 10^{-4}$ (Bonati et al., 2015 [1512.01544])

Holo. Yang–Mills: $b_4 \simeq 0.033/N^4$ (using $\bar{b}_2^{(\text{lattice})}$) (Bigazzi et al., 2015 [1506.03826])
$\longrightarrow b_4^{(\text{HYM})}(N=2) \sim 2 \cdot 10^{-3} \sim 3.3\, b_4^{(\text{lattice})}(N=2)$

Euclidean topological charge density 2-pnt correlator

$G(x) = \int d^4x \, \langle q(x)q(0)\rangle < 0$ for any $r \equiv |x| > 0$ because of **reflection positivity**.

Reflection positivity: $\langle \Theta[\mathcal{O}(\Theta x)]\mathcal{O}(x)\rangle > 0$ for any operator $\mathcal{O}$
$\Theta =$ Euclidean time reflection + complex conjugation

Since $q(x)$ is T and P odd $\implies G(x) < 0$ for $r > 0$.

- Perturbation theory: $G(x) \sim -C^2/(r^8 \log^2 r)$ for short distances $r \ll 1$ (Vicari, 1998 [hep-lat/9901008])

- $G(x) \sim -A\exp\{-mr\}$ for large distances $r \gg 1$, with m mass of lightest state in the PC $= -+$ channel (JLQCD Collaboration, 2015 [1509.00944])

However: $\int d^4x \, G(x) = \chi = \langle Q^2\rangle / V > 0$. How to reconcile with reflection positivity?

$\implies G(x)$ has a positive non-integrable singularity in $x = 0$ due to a contact term which cancels the negative divergent integral $\lim_{\varepsilon\to 0}\int_{r>\varepsilon} d^4x \, G(x)$. The residue of such cancellation is the positive and finite topological susceptibility.

Similar arguments apply to $\chi' \propto \int d^4x \, G(x)|x|^2$, which is finite too. Due to $|x|^2$ promoting the long-distance tail and depressing the contact term, χ' can be either positive or negative, and does not vanish in the chiral limit (unlike χ).