Localisation of Dirac modes in finite-temperature $\mathbb{Z}_2$ and $\mathbb{Z}_3$ gauge theories on the lattice

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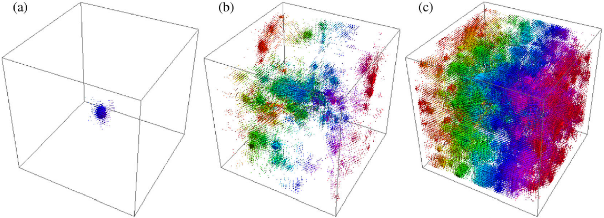
Work done in collaboration with Matteo Giordano

Introduction I

- Localisation of low modes of the Dirac operator was observed in QCD and other gauge theories above the deconfinement transition [Garcia-Garcia and Osborn, 2007, Ujfalusi et al., 2015]



Localisation and delocalisation from Ref. [Ujfalusi et al., 2015]

- Localisation of eigenmodes caused by disorder is a well-studied phenomenon in condensedmatter physics

Introduction II

- Low-lying Dirac modes in QCD at high temperature shares the same critical features with three-dimensional Hamiltonians with on-site disorder
- Sea/island picture $\rightarrow$ Ordered Polyakov loops in deconfined phase (deconfinement transition is the only thing that is needed). In this ordered "sea" modes are localised on the fluctuations of Polyakov loops [Bruckmann et al., 2011]
- To push the connection of these properties to its limit $\rightarrow$ $\mathbb{Z}_2$ gauge theory in 2+1 dimensions and study the spectrum of the staggered Dirac operator, link variables: $U_\mu(n) = \pm 1$, N_t fixed.

$\mathbb{Z}_2$ and $\mathbb{Z}_3$ gauge theory I

- $A_\mu(x)$ is replaced by parallel transporters $U_\mu(n)$
- The Wilson action reads

$$S[U] = -\beta \sum_n \sum_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{3} U_{\mu\nu}(n),$$

where $U_{\mu\nu} = U_\mu(n) U_\nu(n+\hat{\mu}) U^*_\mu(n+\hat{\nu}) U^*_\nu(n)$

- Expectation value of an observable O is defined by

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \cdot e^{-S[U]} O[U] = Z^{-1} \sum_{[U]} e^{-S[U]} O[U], Z = \sum_{[U]} e^{-S[U]}$$

- The staggered Dirac operator reads

$$D_{nn'} = \frac{1}{2}\sum_{\mu=1}^{3} \eta_\mu(n)(U_\mu(n)\delta_{n+\hat{\mu},n'} - U^*_\mu(n-\hat{\mu})\delta_{n-\hat{\mu},n'}),$$

where $\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu}$.

- $D_{n,n'}$ is anti-Hermitian:

$$D_{n,n'}\psi_l = i\lambda_l\psi_l, \quad \lambda_l \in \mathbb{R}$$

- $\{\varepsilon, D\} = 0$ with $\varepsilon(n) = (-1)^{\sum_{\mu=1}^{3} n_\mu}$ implies

$$D\varepsilon\psi_l = -i\lambda_l\varepsilon\psi,$$

so the spectrum is symmetric about zero.

- The deconfinement transition is signalled by breaking of the symmetry

$$U_1(N_t - 1, \vec{x}) \to zU_1(N_t - 1, \vec{x}),\ \forall \vec{x}\ \ z \in \mathbb{Z}_n,$$

the relevant order parameter is the Polyakov loop. $\to$ "physical" and "unphysical" sectors

Sea/islands picture

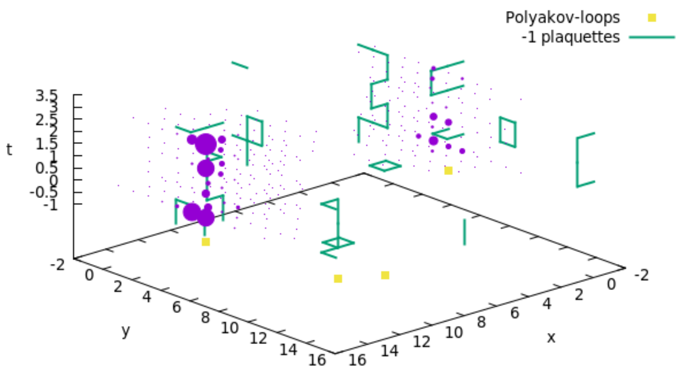
- The Polyakov loop at $\vec{x} \to P(\phi(\vec{x}))$ with some phase $\phi(\vec{x})$.
- Assuming trivial spatial links and temporal links U, where $U^{N_t} = e^{i \cdot \phi}$ the eigenvalues are

$$\lambda = \sqrt{\sin^2 \omega + \sum_j \sin^2 p_j},$$

where $\frac{L p_j}{2\pi} = 0, \ldots, N_s - 1$ and $\omega = \frac{1}{N_t}(\pi + \phi + 2\pi k)$ with $k = 0, \ldots, N_t - 1$

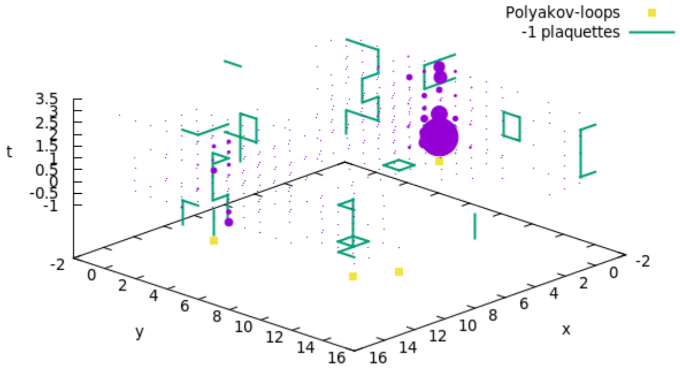
- The lowest branch of eigenvalues reads

$$M(\phi) = \sin\left(\frac{\pi - |\phi|}{N_t}\right),$$



$N_t = 4,\ N_s = 16,\ \lambda_1 = 0.008$

Configuration in $\mathbb{Z}_2$ gauge theory



$N_t = 4,\ N_s = 16,\ \lambda_1 = 0.023$

Localisation of eigenmodes of the Dirac operator

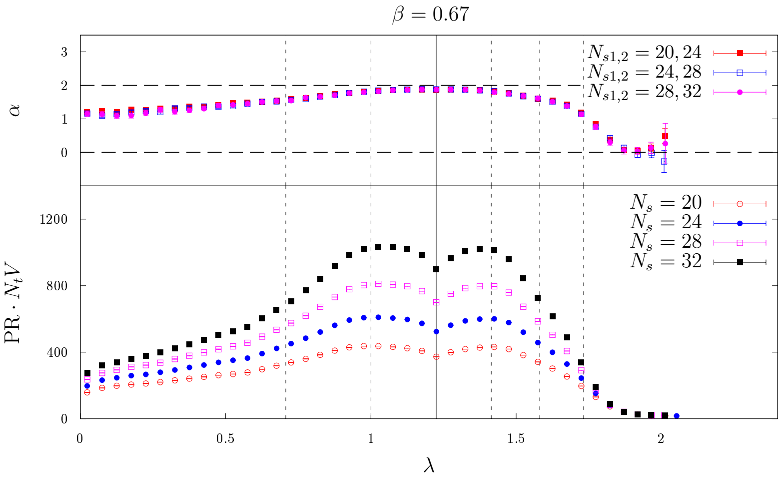
- $\mathrm{IPR}_l = \sum_n |\psi_l(n)|^4$
- $\mathrm{PR}_l{=}\mathrm{IPR}_l^{-1}(N_t V)^{-1}$
- The fractal dimension makes quantitative the scaling with the volume:

$$\mathrm{PR}(\lambda, N_s) \approx c(\lambda) N_s^{\alpha(\lambda)-2}$$

Localised mode $\rightarrow \alpha = 0$
Delocalised mode $\rightarrow \alpha = 2$

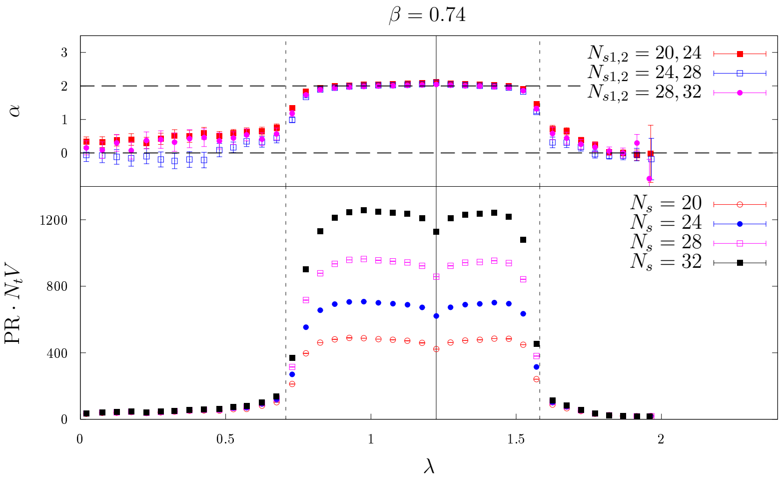
$$\alpha(\lambda) = 2 + \log\left(\frac{\mathrm{PR}(\lambda, N_{s_1})}{\mathrm{PR}(\lambda, N_{s_2})}\right) \Big/ \log\left(\frac{N_{s_1}}{N_{s_2}}\right)$$

Confined phase



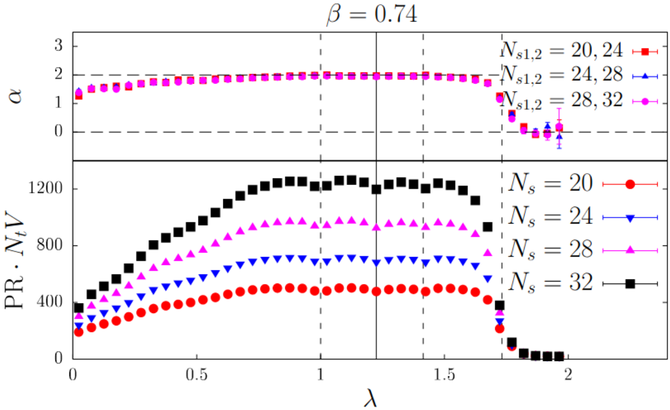
Low modes and bulk modes are not localised, high modes are localised

Deconfined phase ($\overline{P} > 0$)



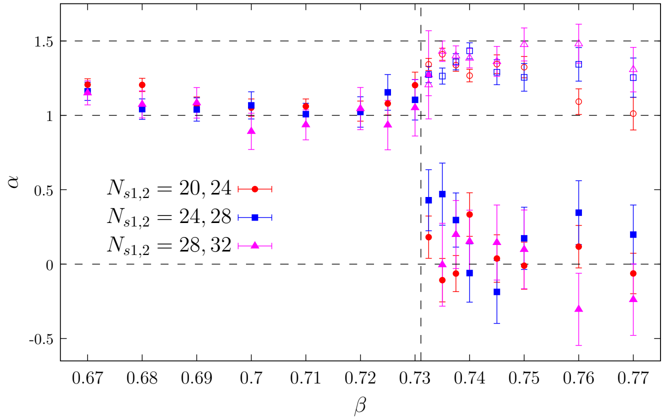
Both low and high modes are localised, bulk modes are delocalised

Deconfined phase ($\overline{P} < 0$)



In the unphysical sector low modes are not localised, high modes remain localised.

Fractal dimension of near zero modes



Fractal dimension drops to zero at the deconfinement transition ($\beta_c(N_t = 4) = 0.73107(2)$ [Caselle and Hasenbusch, 1996])

Sea/island picture of localisation

- How much of the wave function is localised on negative Polyakov loops?

$$\mathscr{P} = \sum_{x,t} P(x)|\psi(x,t)|^2$$

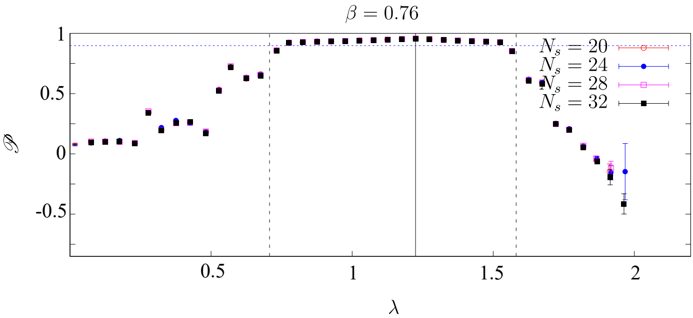
- Delocalised modes:

$$\mathscr{P} \approx \frac{1}{VN_t} \sum_{x,t} P(x) = \frac{1}{V} \sum_{x} P(x) = \overline{P}$$

- Localised modes:

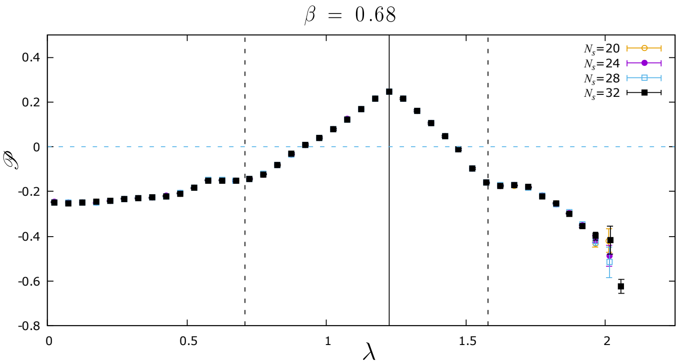
$$\mathscr{P} \approx \sum_{(x,t)\in V_0} P(x)|\psi(x,t)|^2 \approx \overline{P}_{V_0}$$

Sea/island picture of localisation, deconfined phase



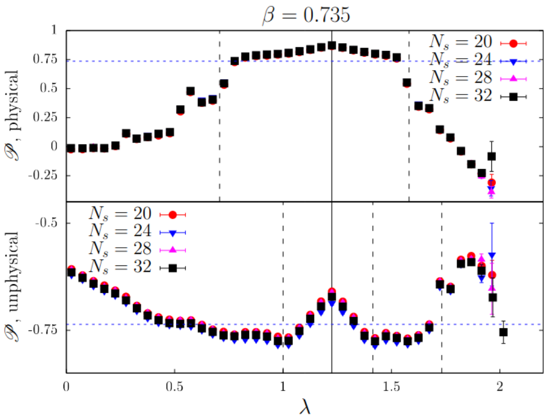
For delocalised modes $\mathscr{P}$ takes the value of the average Polyakov loops. However, for localised modes $\mathscr{P}$ drops significantly

Sea/island picture of localisation, confined phase



For localised high modes $\mathscr{P}$ becomes much lower, while for delocalised modes $\mathscr{P}$ is closer to $\overline{P}$

Sea/island picture of localisation, deconfined phase



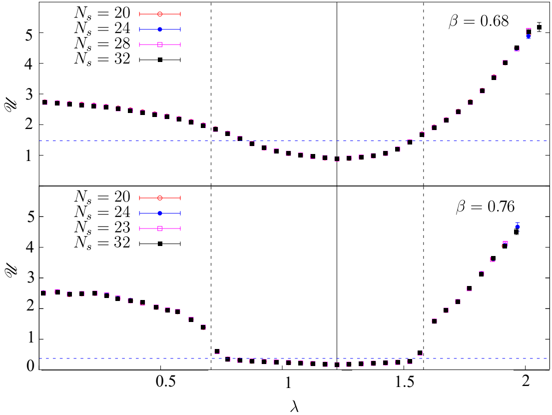
Unphysical and physical sector, $\mathscr{P}$ comparison in the deconfined phase

Localisation and negative plaquettes

Plaquettes encode dynamics. How do localised modes correlate with negative plaquettes?

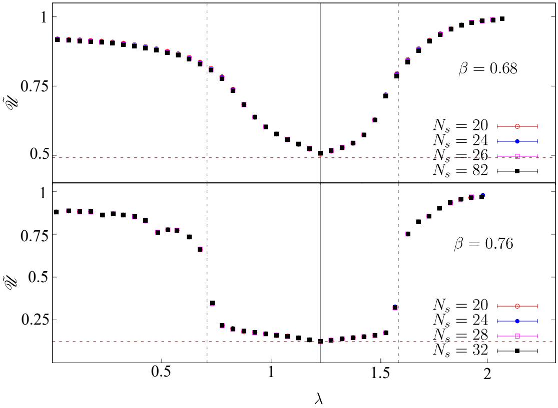
$$A(n) = \frac{1}{2} \sum_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{3} [4 - U_{\mu\nu}(n) - U_{\mu\nu}(n-\hat{\mu}) - U_{\mu\nu}(n-\hat{\nu}) - U_{\mu\nu}(n-\hat{\mu}-\hat{\nu})]$$

- $\mathscr{U} = \sum_n A(n)|\psi(n)|^2$ measures the average number of negative plaquettes touched by the modes
- $\widetilde{\mathscr{U}} = \sum_{A(n)>0,n} |\psi(n)|^2$ measures how much of the modes lives on sites touched by negative plaquettes



Localised modes prefer to live in clusters of negative plaquettes

Localisation and negative plaquettes



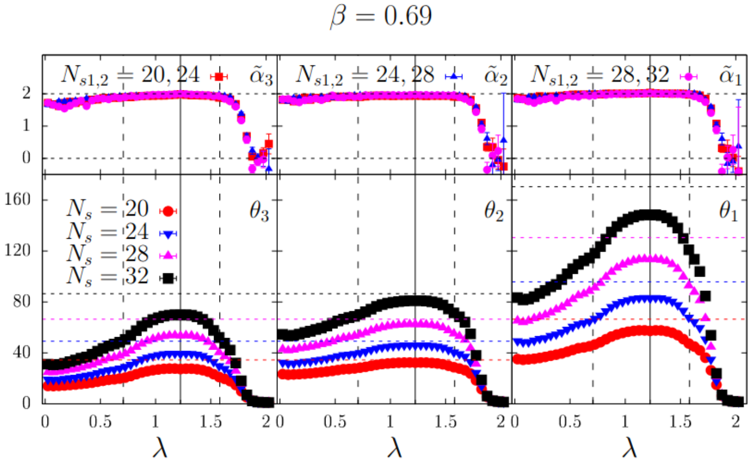
For localised modes most part of the modes live on sites that are touched by at least one negative plaquette

- The shape of eigenmodes was studied by the inertia tensor

$$\Theta_{\mu\nu} = \sum_n |\psi(n)|^2 \left[\delta_{\mu\nu} \sum_\rho (n_\rho - \overline{n}_\rho)_P^2 - (n_\mu - \overline{n}_\mu)_P (n_\nu - \overline{n}_\nu)_P\right]$$

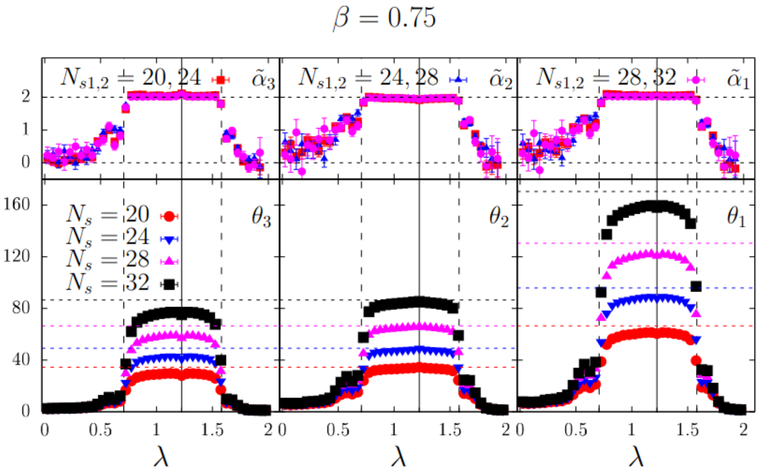
- $\theta_1 \geq \theta_2 \geq \theta_3$ are the eigenvalues
  - $\theta_1 > \theta_2 \approx \theta_3 \quad \rightarrow$ oblate shape
  - $\theta_1 \approx \theta_2 > \theta_3 \quad \rightarrow$ prolate shape
  - $\theta_1 \approx \theta_2 \approx \theta_3 \quad \rightarrow$ spherical modes
- The orientation of the mode is also obtainable compared to the temporal direction $\rightarrow \quad \varphi_1, \varphi_2, \varphi_3$
- $\theta_i \sim N_s^{\tilde{\alpha}_i}$

Inertia tensor of the eigenmodes-Confined phase



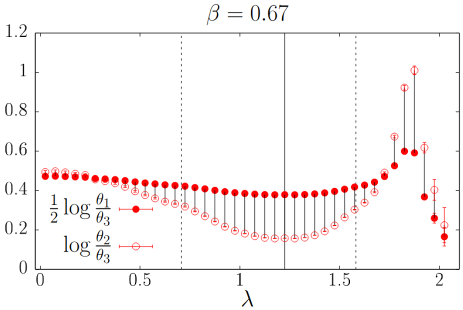
High modes are localised in all directions.

Inertia tensor of the eigenmodes-Deconfined phase



Both low and high Dirac modes are localised.

Inertia tensor of the eigenmodes-Shape of modes

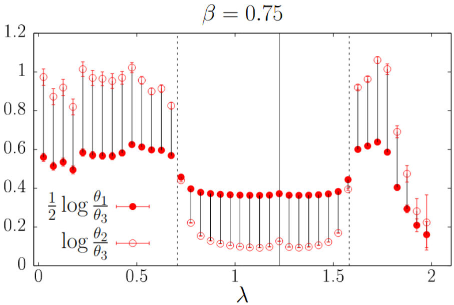


The quantity

$$\log\frac{\theta_2}{\theta_1} - \frac{1}{2}\log\frac{\theta_1}{\theta_3} = \frac{1}{2}\log\frac{\theta_2^2}{\theta_1\theta_3}$$

measures the prolateness/oblateness of modes.

Inertia tensor of the eigenmodes-Shape of modes



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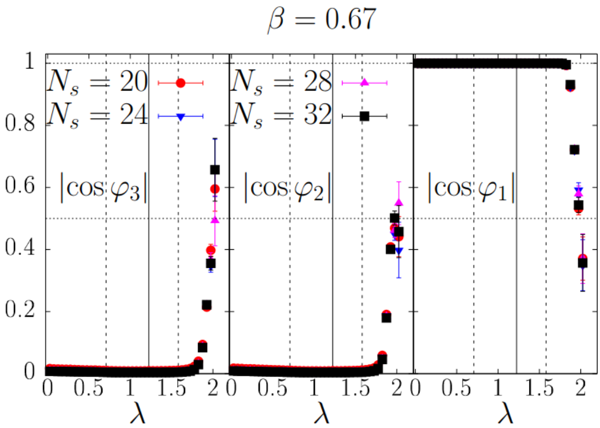
Inertia tensor of the eigenmodes-Orientation of modes-Confined phase



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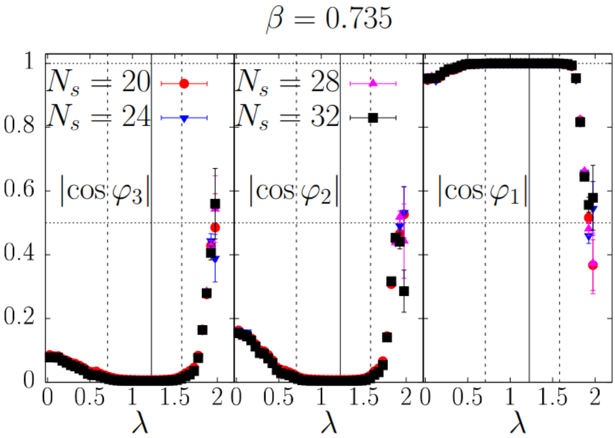

Nontrivial for low modes.

$\mathbb{Z}_3$ gauge theory

- The staggered Dirac operator reads

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  where $\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu}$.
- The lowest branch of eigenvalues reads

$$M(\phi) = \sin\left(\frac{\pi - |\phi|}{N_t}\right),$$

  where now $\pi - \phi > \frac{\pi}{3}$
- Firs task $\rightarrow$ finding β_c

Dual lattice

- $\mathbb{Z}_n$ gauge theories $\rightarrow U_l = e^{\frac{2\pi i \theta_l}{N}} \rightarrow S = \sum_p \left(1 - \cos\left(\frac{2\pi\theta_p}{N}\right)\right)$
- By Fourier-expanding each plaquettes and summing over configurations one can recast the partition function in term of spin-like variables
- We get the clock model

$$E = -\beta^* \sum \cos(2\pi(\theta_i - \theta_j)/n)$$

- In the Potts-model

$$E = -\beta^{**} \sum \delta_{s_i s_j}$$

-

$$\beta^* = \frac{2}{3} \log\left(\frac{e^{\beta} - 2e^{-\frac{\beta}{2}}}{e^{\beta} - e^{-\frac{\beta}{2}}}\right)$$

Binder cumulant and phase transition

- Correlation length

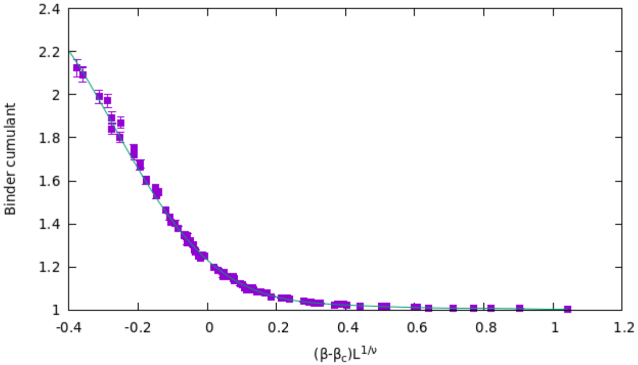
$$\xi \sim |\beta - \beta_c|^{-\nu}$$

- Near the transition

$$\phi(L, \beta - \beta_c) = \phi\left(\frac{\xi}{L}\right) = \tilde{\phi}(|\beta - \beta_c|L^{\frac{1}{\nu}})$$

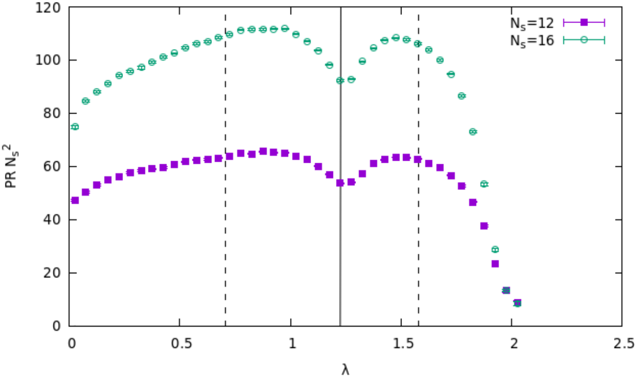
- Then $\phi \approx F_0 + F_1|\beta - \beta_c|L^{1/\nu} + \ldots$
- I used the binder cumulant $\sim \frac{\langle\phi^2\rangle^2}{\langle\phi\rangle^4}$

Binder cumulant



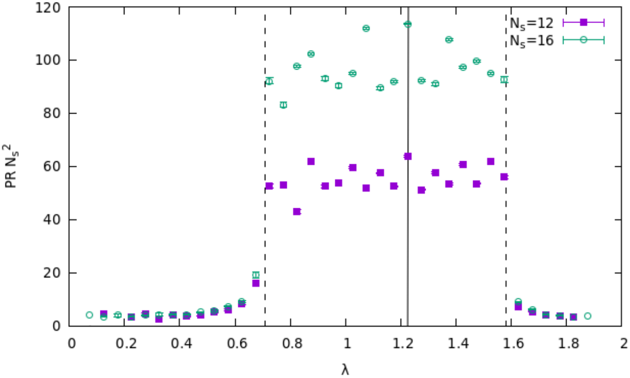
$\rightarrow \beta^{**} = 0.5642(1) \rightarrow \beta = 1.067(2)$

Size of modes-Confined phase



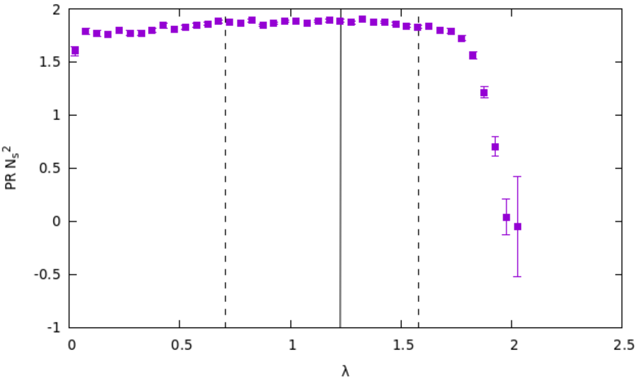
Only high modes are localised

Size of modes-Deconfined phase



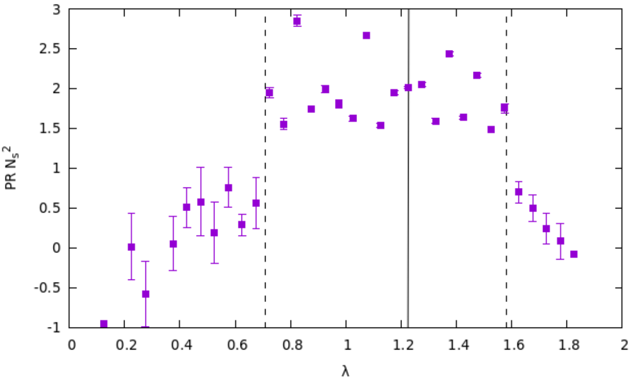
Localised high and low modes

Fractal dimension-Deconfined phase



$$\mathrm{PR} \propto N_s^{\alpha-2}$$

Fractal dimension-Deconfined phase



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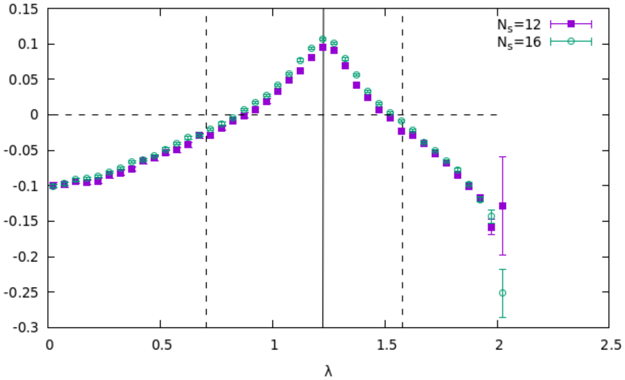
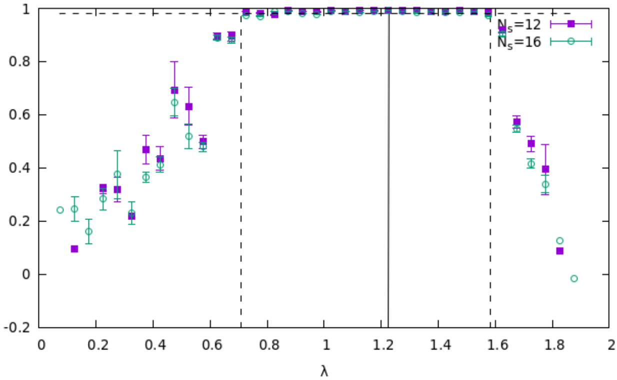
Sea/islands picture



$$\mathscr{P} = \sum P|\psi|^2$$

Sea/islands picture



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Conclusion

- Localisation of low modes is present in QCD and many gauge theories, even in $\mathbb{Z}_2$ which is the simplest gauge theory showing a deconfinement transition, and $\mathbb{Z}_3$ gauge theory, where potential well are less favourable

Conclusion

- Localisation of low modes is present in QCD and many gauge theories, even in $\mathbb{Z}_2$ which is the simplest gauge theory showing a deconfinement transition, and $\mathbb{Z}_3$ gauge theory, where potential well are less favourable
- Numerical results confirm the predictions of the sea/island picture of localisation in both theory

- Localisation of low modes is present in QCD and many gauge theories, even in $\mathbb{Z}_2$ which is the simplest gauge theory showing a deconfinement transition, and $\mathbb{Z}_3$ gauge theory, where potential well are less favourable
- Numerical results confirm the predictions of the sea/island picture of localisation in both theory
- A novel result is that the very high modes are localized in both phases of the theory

References I

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Size of modes-Deconfined phase-Unphysical sector