

The full analytical trans-series in integrable field theories

Zoltán Bajnok

**Wigner Research Centre for Physics
Budapest, Hungary**



In collaboration with: M.C. Abbott, J. Balog, A. Hegedus, S. Sadeghian, I. Vona

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2212.09416

Perturbation theory

physical observable

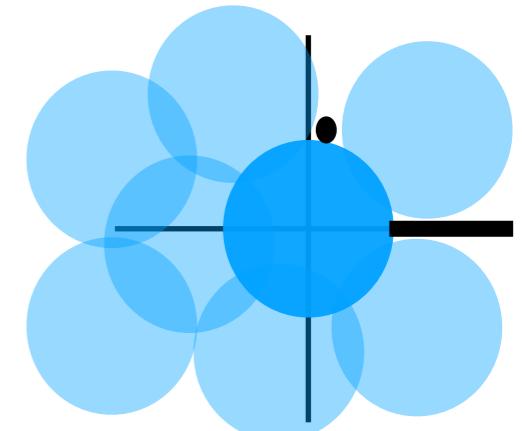
perturbative coefficients

coupling constant

$$\mathcal{O}(g) = \sum_n \alpha_n g^n$$

1. convergent, radius of convergence

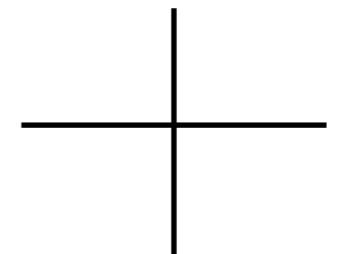
$$r = \lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}}$$



2. asymptotic, radius of convergence is zero

typically in asymptotically free QFTs, such as in QCD,

$$\frac{1}{n} = \lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}}$$



factorial growth

$$\alpha_n \sim n!$$

signals non-perturbative
terms

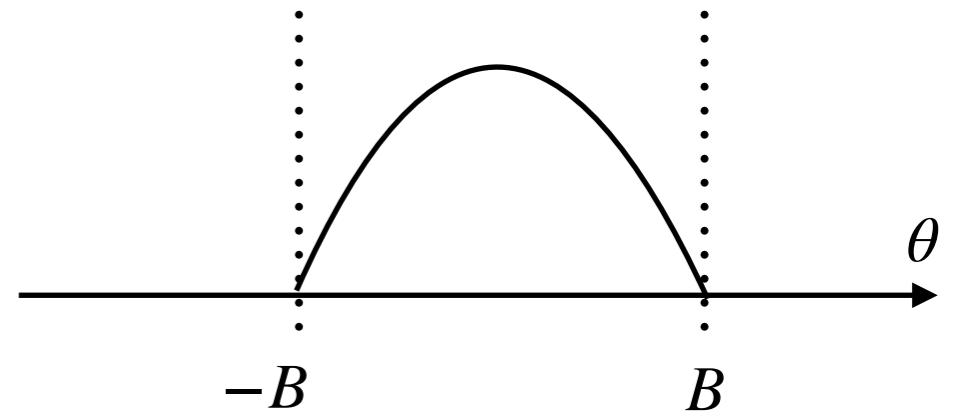
$$e^{-\frac{A}{g}}$$

proliferation of Feynmann diagrams: instantons
specific type of diagrams: renormalons

Can we determine all the non-perturbative corrections from the perturbative ones?

analyse this question in toy models

The mathematical challenge



the integral equation for the unknown $\chi_\alpha(\theta, B)$

$$\chi_\alpha(\theta, B) - \int_{-B}^B K(\theta - \theta') \chi_\alpha(\theta', B) d\theta' = \cosh(\alpha\theta)$$

The observable

$$\mathcal{O}_{\alpha,\beta}(B) = \int_{-B}^B \cosh(\beta\theta) \chi_\alpha(\theta, B) \frac{d\theta}{2\pi}$$

the small parameter

$$B^{-1} \quad e^{-B} \quad (\log B)$$

trans-series

$$\mathcal{O}_{\alpha,\beta}(B) = \sum_n e^{-nBA} \sum_m c_{n,m} B^{-m}$$

Aim: calculate $c_{n,m}$

differential equations

$$\frac{d\mathcal{O}_{\alpha,\beta}}{dB} = \dot{\mathcal{O}}_{\alpha,\beta} = \frac{1}{\pi} \chi_\alpha(B, B) \chi_\beta(B, B)$$

$$\frac{\ddot{\chi}_\alpha(B, B)}{\chi_\alpha(B, B)} - \alpha^2 = f(B)$$

basic observables

$$\mathcal{O}_{1,1} \rightarrow \chi_1 \rightarrow f \rightarrow \chi_\alpha \rightarrow \mathcal{O}_{\alpha,\beta}$$

Related problems

$$\chi_\alpha(\theta, B) - \int_{-B}^B K(\theta - \theta') \chi_\alpha(\theta', B) d\theta' = \cosh(\alpha\theta)$$

2 dimensional integrable QFTs coupled to a conserved charge

$$\mathcal{O}_{\alpha,\beta}(B) = \int_{-B}^B \cosh(\beta\theta) \chi_\alpha(\theta, B) d\theta$$

Energy density

$$\mathcal{O}_{1,1}(B) = \epsilon(B)$$

Density

$$\mathcal{O}_{1,0}(B) = \rho(B)$$

$$K(\theta) = -\frac{i}{2\pi} \partial_\theta \log S(\theta)$$

scattering matrix

O(N) sigma models

$$S(\theta) = -\frac{\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi}) \Gamma(\Delta - \frac{i\theta}{2\pi}) \Gamma(1 + \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} + \Delta + \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi}) \Gamma(\Delta + \frac{i\theta}{2\pi}) \Gamma(1 - \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} + \Delta - \frac{i\theta}{2\pi})}$$

$$\Delta^{-1} = N - 2$$

N=1 SUSY O(N) sigma models

O(N) Gross-Neveu model

SU(N) principal chiral field, with various charges

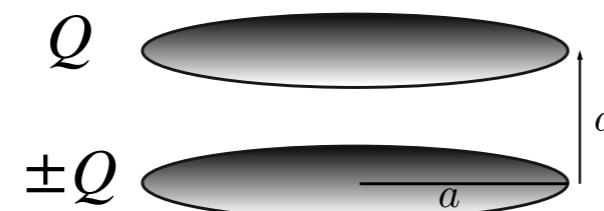
SU(2)=O(4) sigma model

O(3) model

$$K(\theta) = \frac{1}{\theta^2 + \pi^2}$$

Lieb-Liniger model **Gaudin-Yang model**

coaxial disk capacitor



$$B = \frac{a}{d}$$

$$\mathcal{O}_{0,0}(B)$$

capacity

Plan

- Definition of the 2D $O(N)$ models in a magnetic field
 - Integrable description, Thermodynamic Bethe Ansatz (TBA)
 - Expansion of the TBA, perturbative coefficient c_n
-
- Analytic structure on the Borel plane from asymptotic c_{2000}
-
- median resummation, non-perturbative contributions, trans-series
 - analytic considerations
-
- resurgence and trans-series in the $O(3)$ model, instantons
-
- Solution based on the Wiener-Hopf and full analytic trans-series
-
- Conclusions, outlook

Definition of the O(N) (non-linear) sigma model

N scalar fields in 2D living on the unit sphere

$$\Phi_1^2 + \dots + \Phi_N^2 = 1$$

magnetic field is coupled the conserved O(N) charge

$$\mathcal{L} = \frac{1}{2\lambda^2} \left\{ \partial_\mu \Phi_i \partial^\mu \Phi_i + 2ih(\Phi_1 \partial_0 \Phi_2 - \Phi_2 \partial_0 \Phi_1) + h^2(\Phi_3^2 + \dots + \Phi_N^2 - 1) \right\}$$

bare coupling

Euclidean Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 - hQ_{12}$$

Perturbation theory

$$\Phi_1^2 = 1 - \lambda^2(\varphi_2^2 + \dots + \varphi_N^2)$$

$$\lambda \varphi_i = \Phi_i$$

$$e^{-V\mathcal{F}(h)} = \int \mathcal{D}^{N-1}[\varphi] e^{-\int d^D x \mathcal{L}(x)}$$

$$D = 2 - \epsilon$$

dimensional regularisation

Legendre transformation

$$\rho = -\partial \mathcal{F}/\partial h,$$

density

groundstate energy

$$\epsilon(\rho) = \mathcal{F}(h) - \mathcal{F}(0) + \rho h$$

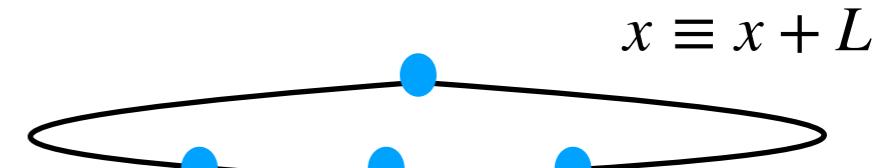
very hard

Groundstate energy density in integrable models

**multiparticle state on the circle
momentum quantization**

$$p = m \sinh \theta$$

$$p = m\theta$$



Thermodynamic limit of the Bethe Ansatz: TBA

$$\chi(\theta) - \int_{-B}^B K(\theta - \theta') \chi(\theta') d\theta' = m \cosh \theta$$

density

$$\rho(B) = \int_{-B}^B \frac{d\theta}{2\pi} \chi(\theta)$$

ground state energy

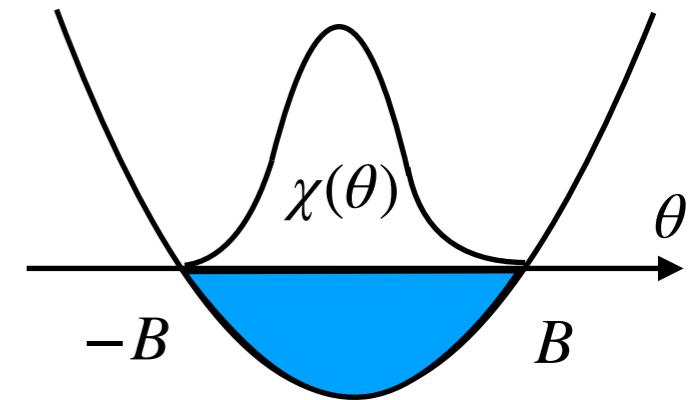
$$\epsilon(B) = m \int_{-B}^B \frac{d\theta}{2\pi} \cosh \theta \chi(\theta)$$

$$2\pi K(\theta) = - i \partial_\theta \log S(\theta)$$

$$e^{ip_j L} \prod_k S(\theta_j - \theta_k) = 1$$

$$E = m \cosh \theta$$

$$E = m \frac{\theta^2}{2}$$



Integrable QFTs in a magnetic field coupled to a conserved charge $\mathcal{H} = \mathcal{H}_0 - hQ$

particles charged under

Q **condense into the vacuum**

$$E_\pm = m \cosh \theta \pm h$$

Polyakov, Wiegmann, *Phys.Lett.B* 131 (1983) 121

$$S(\theta) = - \frac{\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi}) \Gamma(\Delta - \frac{i\theta}{2\pi}) \Gamma(1 + \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} + \Delta + \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi}) \Gamma(\Delta + \frac{i\theta}{2\pi}) \Gamma(1 - \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} + \Delta - \frac{i\theta}{2\pi})}$$

Hasenfratz, Maggiore, Niedermayer, *Phys.Lett.B* 245 (1990) 522

O(N) nonlinear sigma model

**Asymptotically free
dynamically generated scale**

Large B expansion of the TBA

Volin, Phys. Rev. D 81 (2010) 105008 · 0904.2744

The resolvent

$$R(\theta) = \int_{-B}^B \frac{d\theta'}{2\pi} \frac{\chi(\theta')}{\theta - \theta'}$$

density: residue at ∞

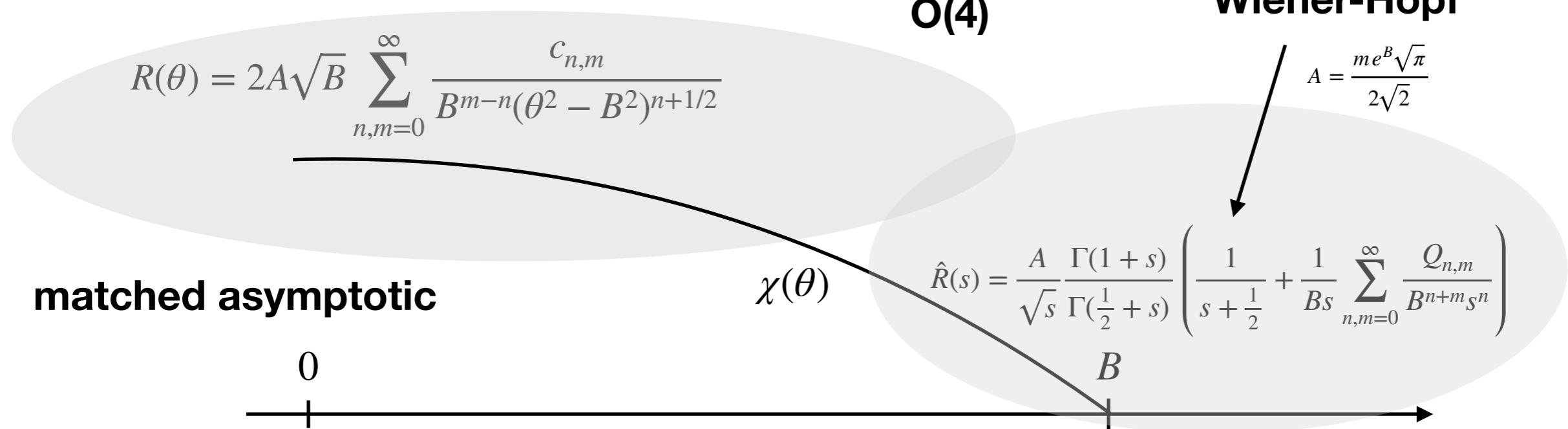
perturbative expansion in $1/B$

its Laplace transform

$$\hat{R}(s) = \int_{-i\infty+0}^{i\infty+0} \frac{dz}{2\pi i} e^{sz} R(B + z/2)$$

$$z = 2(\theta - B)$$

energy: $\frac{\epsilon}{m} = \int_{-B}^B \cosh \theta \chi(\theta) \frac{d\theta}{2\pi} = \frac{e^B}{4\pi} \hat{R}(1/2)$



$$\rho = A \frac{\sqrt{B}}{\pi} \hat{\rho} = A \frac{\sqrt{B}}{\pi} \sum_{m=0}^{\infty} c_{0,m} B^{-m}$$

$$\epsilon = \frac{me^B A}{4\sqrt{2\pi}} (1 + \sum_{k=0}^{\infty} \epsilon_k B^{-k-1}) \quad \epsilon_k = \sum_{j=0}^k 2^{j+1} Q_{j,k-j}$$

Perturbative coefficients

density $\hat{\rho}(B) = 1 + \sum_{n=1}^{\infty} \frac{u_n}{B^n}$ $u_1 = -\frac{3}{8} + \frac{a}{2}$ $u_2 = -\frac{15}{128} + \frac{3a}{16} - \frac{a^2}{8}$ $u_3 = \frac{3\zeta_3}{64} + \frac{a^3}{16} - \frac{9a^2}{64} + \frac{45a}{256} - \frac{105}{1024}$

$a = \ln 2$ odd zeta functions

energy $\hat{\epsilon}(B) = 1 + \sum_{n=1}^{\infty} \frac{\xi_n}{B^n}$ $\left\{ \frac{1}{4}, \frac{9}{32} - \frac{a}{4}, \frac{a^2}{4} - \frac{9a}{16} + \frac{57}{128}, -\frac{a^3}{4} + \frac{27a^2}{32} - \frac{171a}{128} - \frac{27\zeta_3}{256} + \frac{1875}{2048}, \dots \right\}$

Volin, *Phys.Rev.D* 81 (2010) 105008 • e-Print: [0904.2744](#)

Marino, Reis, *JHEP* 04 (2020) 160 • e-Print: [1909.12134](#)

22 coefficients

44 coefficients

50 coefficients

free energy in the running coupling

$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2} \log B \alpha = \log 2 \hat{\rho}$$

$$f(\alpha) = \frac{\epsilon}{\rho^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \chi_n \alpha^n = \frac{\pi}{2} \left(1 + \frac{\alpha}{2} + \frac{\alpha^2}{4} + \frac{10 - 3\zeta_3}{32} \alpha^3 + \chi_5 \alpha^4 + \dots \right)$$

$$\left\{ 1, \frac{1}{2}, \frac{1}{4}, \frac{5}{16} - \frac{3\zeta_3}{32}, \frac{53}{96} - \frac{9\zeta_3}{64}, -\frac{189\zeta_3}{512} - \frac{405\zeta_5}{2048} + \frac{487}{384}, \dots \right\}$$

Hasenfratz, Maggiore, Niedermayer, *Phys.Lett.B* 245 (1990) 522

Comparing ordinary perturbation theory in $\frac{h}{\Lambda}$ to expansion of TBA in $\frac{h}{m}$
relation between mass and scale can be obtained $m/\Lambda = (8/e)^{\Delta}/\Gamma(1 + \Delta)$

Numerical data for O(4)

density

$$\hat{\rho}(B) = 1 + \sum_{n=1}^{\infty} \frac{u_n}{B^n}$$

2000 coefficients for 7000 digits

energy

$$\hat{\epsilon}(B) = 1 + \sum_{n=1}^{\infty} \frac{\xi_n}{B^n}$$

2000 coefficients for 7000 digits

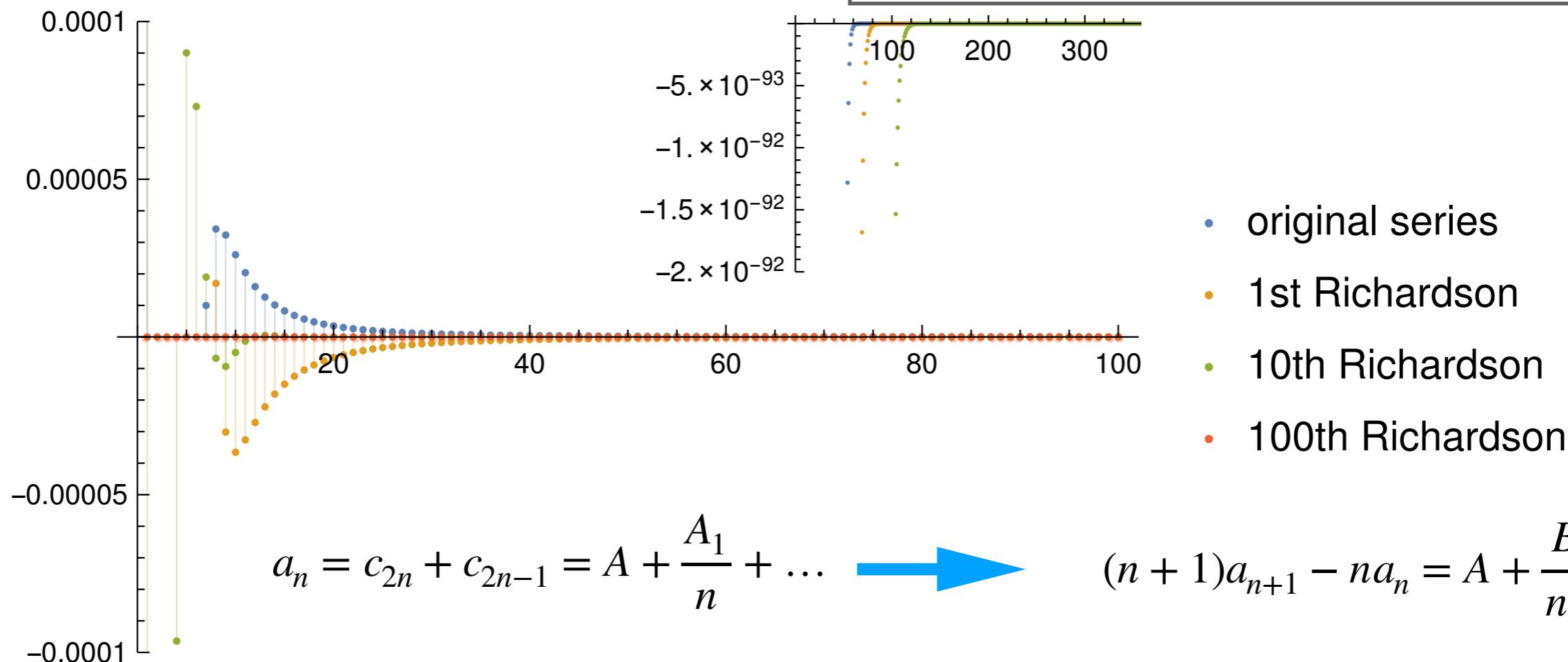
free energy

$$\hat{f}(\alpha) = \frac{\hat{\epsilon}}{\hat{\rho}^2} = \sum_{n=1}^{\infty} \chi_n \alpha^n$$

1400 coefficients for 4000 digits

factorial growth:

$$c_n = \frac{\chi_{n+2} 2^{n+1}}{\Gamma(n+1)} = p^+ + (-1)^n p^- + \frac{1}{n} (\dots)$$



Asymptotic behaviour

perturbative coefficient grow factorially:
how to give meaning to the series?

$$c_n = \frac{\chi_{n+2} 2^{n+1}}{\Gamma(n+1)}$$

Borel function

$$\Psi(t) = \sum_{n=1}^{\infty} c_n t^n$$

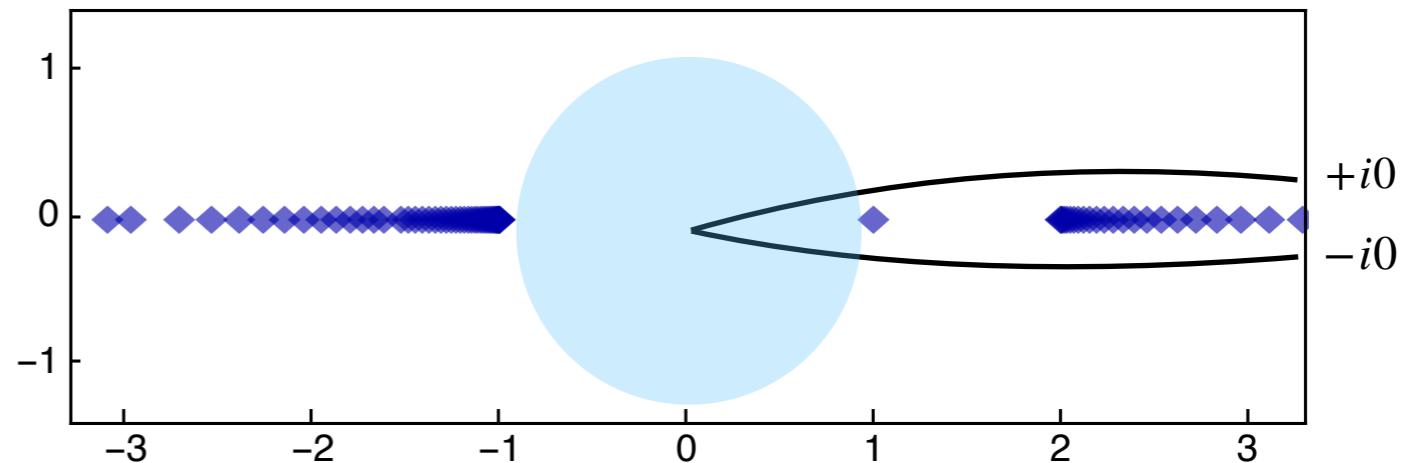
path integral

$$e^{-V\mathcal{F}(h)} = \int \mathcal{D}^3[\varphi] e^{-S[\varphi]} = \int_0^{\infty} dt e^{-\frac{2t}{\alpha}} \int_{S[\varphi]=\frac{2t}{\alpha}} \mathcal{D}^3[\varphi] e^{-S[\varphi]}$$

convergence radius 1

Pade approximant

$$\Psi(t) \approx \frac{\sum_{i=1}^n \beta_i t^i}{1 + \sum_{j=1}^m \gamma_j t^j}$$



Ambiguity from the pole

$$\frac{i\pi^2 \alpha \text{res}_1 \Psi(t)}{4} e^{-\frac{2}{\alpha}}$$

$$f^{(\pm)}(\alpha) = \frac{\pi}{2} \left[\chi_1 \alpha + \chi_2 \alpha^2 + \alpha \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}} \Psi(t) dt \right]$$

Better approximation \longrightarrow

Conformal mapping

Conformal mapping vs numerical solution



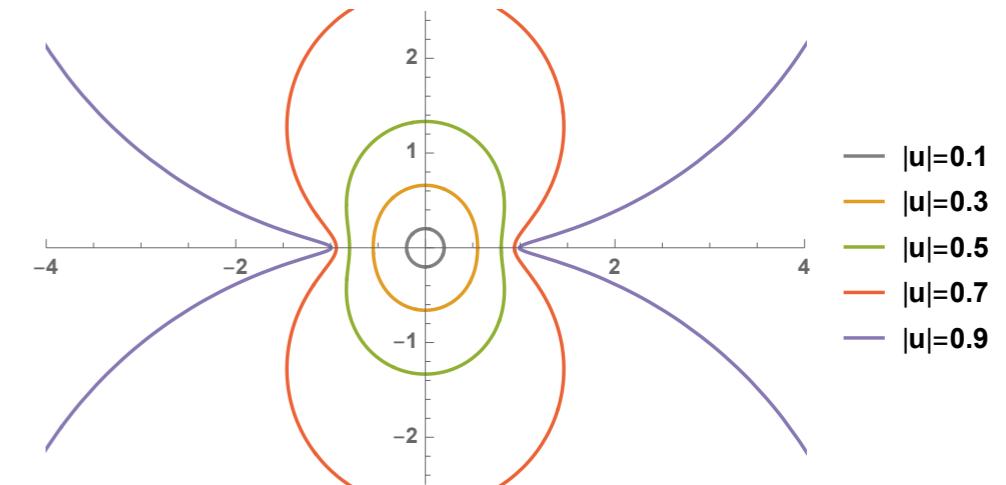
Conformal mapping maps the t-plane to the unit u-disk

$$u(t) = \frac{1 - \sqrt{1 - t^2}}{t}$$

$$\tilde{\Psi}(u) = \sum_{n=1}^{\infty} b_n u^n$$

$$\tilde{\Psi}(t) = \sum_{n=1}^{\infty} b_n u(t)^n$$

$$f^{(\pm)}(\alpha) = \frac{\pi}{2} \left[\chi_1 \alpha + \chi_2 \alpha^2 + \alpha \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}} \tilde{\Psi}(t) dt \right]$$



**Compare to numerical solution
in a Chebyshev basis**

$$\chi(\theta) = \sum_{j=1}^{(n_c+1)/2} s_j T_{2j-2}(\theta/B), \quad T_n(x) = \cos(n \arccos x)$$

precision=30 digits

$$\operatorname{Im}(f^{(+)}(\alpha)) = \alpha c_0 e^{-2/\alpha} + \alpha e^{-4/\alpha} (c_1 + c_2 \alpha + \dots)$$

$$c_0 = 1.70067333(1)$$

$$c_1 = -1.70067333(1)$$

$$c_2 = 0.637752(1)$$

$$c_3 = -0.1727(1)$$

$$\operatorname{Re}(f^{(+)}(\alpha)) = f_{\text{TBA}}(\alpha) + \alpha e^{-8/\alpha} (d_1 + d_2 \alpha + \dots)$$

$$d_1 = -0.9206(1)$$

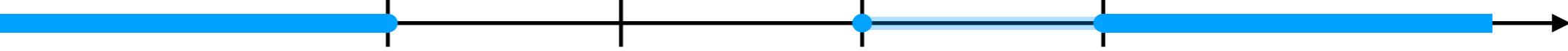
$$d_2 = 0.575(3)$$

Asymptotic analysis

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} s_n / z^n \quad z = 2B \quad c_n = s_{n+1} / n! \quad \Psi(t) = \sum_{n=0}^{\infty} c_n t^n$$

Asymptotic large n behaviour

$$c_n = (-1)^n \left(p^- + \frac{p_0^-}{n} + \frac{p_1^-}{n(n-1)} + \dots \right) + \left(p^+ + \frac{p_0^+}{n} + \frac{p_1^+}{n(n-1)} + \dots \right) + 2^{-n} \left(q^+ + \frac{q_0^+}{n} + \frac{q_1^+}{n(n-1)} + \dots \right)$$

$\log(1+t)[-p_0^- + p_1^-(1+t) + \dots] + \frac{p^-}{1+t}$ $\frac{p^+}{1-t}$


$\log(1+t)\bar{\Psi}(1+t)$

alien derivative

$$\Delta_{\pm 1}\Psi(z) = \mp i2\pi \left\{ p^{\pm} \pm \sum_{m=0}^{\infty} \frac{(\pm 1)^m p_m^{\pm}}{z^{m+1}} \right\}$$

Asymptotics for $\hat{f}(\alpha)$

numerical fitting

2.80308535473939142809960724226717498614747943851074832268840733301275 7308679469635279683810414002887

$$p^- = -\frac{e}{8\pi}$$

for 150 digits

$$p_0^- = 0$$

for 147 digits

$$p_1^- = \frac{e}{4\pi}$$

for 144 digits

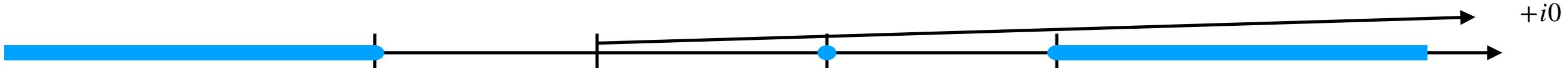
$$p_2^- = \frac{e}{4\pi} \left(-\frac{1}{2} - \frac{3}{4}\zeta_3 \right)$$

<http://wayback.cecm.sfu.ca/projects/EZFace/>

$$\Delta_{-1}\hat{f}$$

$$\Delta_1\hat{f}$$

$$\Delta_2\hat{f}$$



imaginary ambiguity

$$f^{(+)}(\alpha) = \frac{\pi}{2} \left[\alpha \int_0^{\infty+i0} e^{-\frac{2t}{\alpha}} B(t) dt \right]$$

$$\Im m(f^{(+)}(\alpha)) = \frac{4\pi}{e^2} \alpha e^{-2/\alpha} + \alpha e^{-4/\alpha} \left(-\frac{4\pi}{e^2} + \frac{3\pi}{2e^2} \alpha - \frac{13\pi}{32e^2} \alpha^2 + \dots \right)$$

$$c_0 = 1.70067333$$

$$c_1 = -1.70067333(1)$$

$$c_2 = 0.637752(1)$$

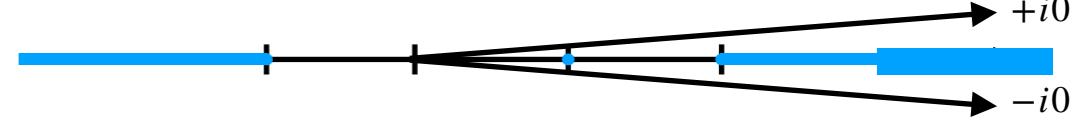
$$c_3 = -0.1727(1)$$

real ambiguity???

Median resummation and Stokes automorphism

$$S_{\pm}(f) = f^{(\pm)} = \chi_1 + \alpha \chi_2 + \int_0^{\infty \pm i0} e^{-tx} \Psi(t) dt$$

$x = \frac{2}{\alpha}$



$$S_+(f) - S_-(f) = -S_+ \left(e^{-x} \Delta_1 f + e^{-2x} \Delta_2 f + \dots + \frac{e^{-2x}}{2} \Delta_1^2 f + \dots \right)$$

$$S_+(f) = S_-(\mathfrak{S}f) \quad ; \quad S_-(f) = S_+(\mathfrak{S}^{-1}f)$$

cut of the cut of the cut...

$$\mathfrak{S} = \exp \left\{ - \sum_{n=1}^{\infty} e^{-nx} \Delta_n \right\}$$

Median resummation

$$S_{\text{med}}(f) = S_-(\mathfrak{S}^{\frac{1}{2}}f) = S_+(\mathfrak{S}^{-\frac{1}{2}}f) = S_+(e^{\frac{1}{2}} \sum e^{-nx} \Delta_n f)$$

$$S_{\text{med}}(f) = S_+ \left(f + \frac{e^{-x}}{2} \Delta_1 f + \frac{e^{-2x}}{2} \Delta_2 f + \dots + \frac{e^{-4x}}{8} \Delta_1 \Delta_3 f + \frac{e^{-4x}}{8} \Delta_2^2 f + \dots \right)$$

$$\Delta_1 f = -\frac{16i}{e^2} \quad \Delta_2 f = \frac{16i}{e^2} \left(1 - \frac{3}{4x} + \frac{13}{32x^2} - \left(\frac{99}{256} - \frac{3}{8} \zeta_3 \right) \frac{1}{x^3} + \dots \right)$$

We need $\Delta_2 \Delta_2 f$

need the asymptotics of $\Delta_2 f$

$$S_+(\Delta_2 f) - S_-(\Delta_2 f) = -S_+ (e^{-2x} \Delta_2 \Delta_2 f) + \dots$$

we found $S_{\text{med}}(f) = \Re(S_+(f)) + \frac{32}{e^4} e^{-8/\alpha} \left(1 - \frac{5\alpha}{8} + \dots \right)$ $d_1 = 0.58607$ $\frac{d_2}{d_1} = -0.6246$

Comparison with TBA

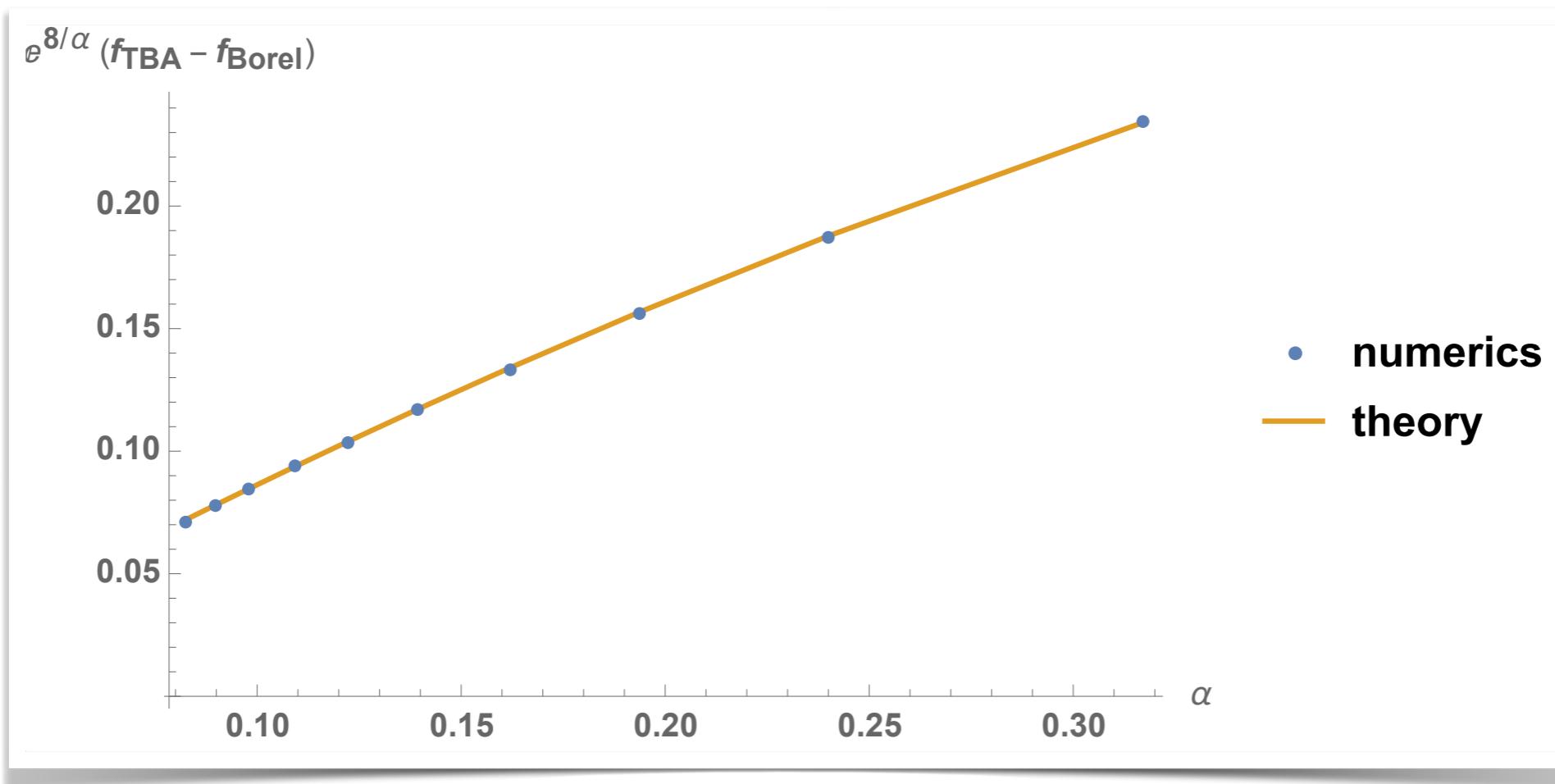
Median resummation

$$S_{\text{med}}(f) = S_+ \left(f + \frac{e^{-x}}{2} \Delta_1 f + \frac{e^{-2x}}{2} \Delta_2 f + \dots + \frac{e^{-4x}}{8} \Delta_1 \Delta_3 f + \frac{e^{-4x}}{8} \Delta_2^2 f + \dots \right)$$

$$f_{\text{Borel}} = S_{\text{med}}(f) = \Re(S_+(f)) + \frac{32}{e^4} e^{-8/\alpha} \left(1 - \frac{5\alpha}{8} + \dots \right)$$
$$d_1 = 0.58607 \quad \frac{d_2}{d_1} = -0.6246$$

$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2} \log B\alpha = \log 2\hat{\rho}$$

We compare to the numerical solution of TBA



Trans-series

Analytic structure of the free energy on the Borel plane



The expansion of the physical observable is a trans-series

$$f_{\text{TBA}} = \sum_{m=0}^{\infty} e^{-\frac{2}{\alpha}m} \sum_{n=1}^{\infty} \chi_n^{(m)} \alpha^{n-1}$$

$$\chi_n^{(0)} = \chi_n$$

perturbative
coefficients

$$e^{-2/\alpha} \chi_n^{(1)}$$

$$\Delta_1 f = -\frac{16i}{e^2}$$

$$e^{-4/\alpha} \chi_n^{(2)}$$

$$\Delta_2 f = \frac{16i}{e^2} \left(1 - \frac{3}{4x} + \frac{13}{32x^2} - \left(\frac{99}{256} - \frac{3}{8}\zeta_3 \right) \frac{1}{x^3} + \dots \right)$$

$$e^{-8/\alpha} \chi_n^{(4)}$$

$$\Delta_2^2 f = \frac{32}{e^4} \left(1 - \frac{5\alpha}{8} + \dots \right)$$

large n asymptotics



large n asymptotics

What is the full trans-series? Is it fixed by the perturbative part?

Asymptotic behaviour in O(3)

perturbative coefficients grow factorially:

$$c_n = \frac{\chi_n 2^{n-1}}{\Gamma(n)}$$

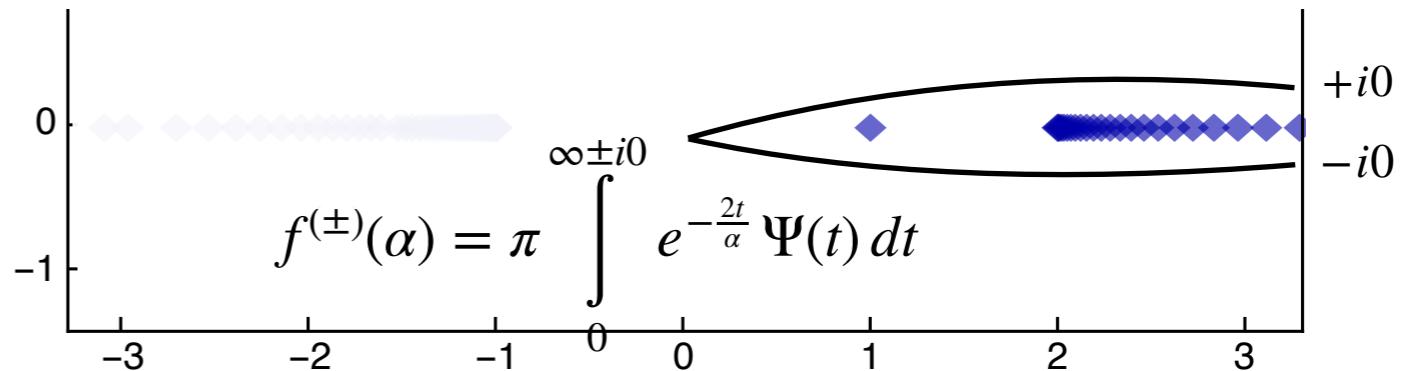
Borel function

$$\Psi(t) = \sum_{n=1}^{\infty} c_n t^n$$

Pade approximant

$$\Psi(t) \approx \frac{\sum_{i=1}^n \beta_i t^i}{1 + \sum_{j=1}^m \gamma_j t^j}$$

improved with conformal mapping



Asymptotic analysis for 336 terms

$$c_n = \frac{8}{e^2} - 2^{-n} a_0 \left(n + a_1 + \frac{a_2}{(n-1)} + \dots \right)$$

$$a_0 = \frac{64}{\pi e^4} \quad \frac{a_1}{a_0} = -\frac{3}{2} \quad \frac{a_2}{a_0} = -\frac{1}{8} \quad \frac{a_n}{a_0} = \Gamma(n+2) \left(b_0 + \frac{b_1}{n} + \dots \right)$$

comparison to numerical solution

$$\Im m(f^+) = \frac{16\pi}{e^2} e^{-\frac{2}{\alpha}} + \frac{e^{-\frac{4}{\alpha}}}{\alpha} \pi \sum_{n=0} a_n 4^{2-n} \alpha^n$$

expected real deviation

$$f_{\text{TBA}} - \Re e(f^+) = e^{-\frac{8}{\alpha}} (b_0 + b_1 \alpha + \dots)$$

real deviation observed

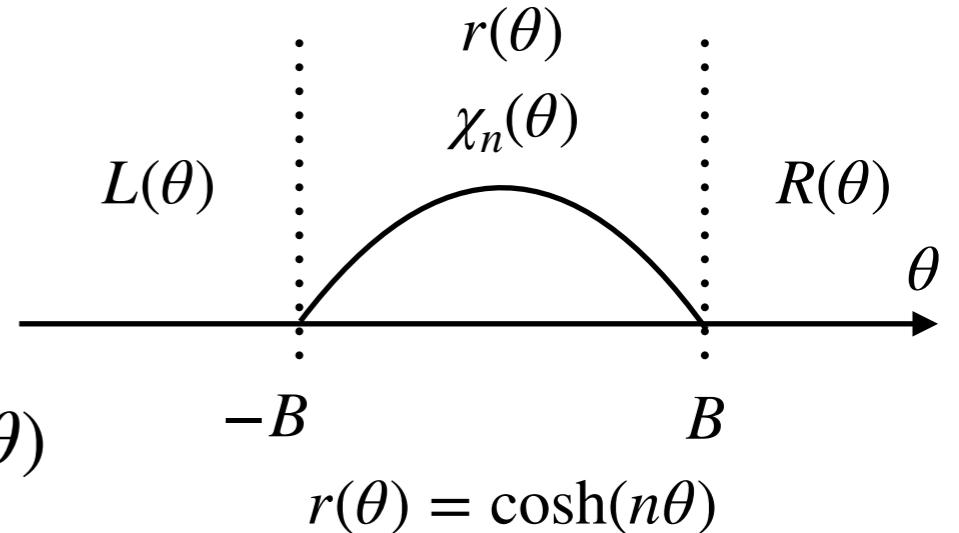
$$f_{\text{TBA}} - \Re e(f^+) = e^{-\frac{2}{\alpha}} A_0 \left(\frac{2}{\alpha} + A_1 + A_2 \log \alpha + A_3 \alpha + \dots \right)$$

Instantons???

Wiener-Hopf solution

1. Extend the integral equation for the whole line

$$\chi_n(\theta) - \int_{-\infty}^{\infty} d\theta' K(\theta - \theta') \chi_n(\theta') = r(\theta) + L(\theta) + R(\theta)$$



2. Use Fourier transformation and invert the kernel

$$(1 - \tilde{K})\tilde{\chi}_n = \tilde{r} + \tilde{L} + \tilde{R}$$

$$\frac{1}{1 - \tilde{K}(\omega)} = G_+(\omega)G_-(\omega)$$

$G_{\pm}(\omega)$ are analytical on the UHP/LHP

$$f_{\pm} = e^{\pm i\omega B} f$$

3. By shifting the functions separate the equations into analytic on the UHP/LHP

$$\frac{\chi_+}{G_+} = G_- r_+ + \alpha(G_+ X_+) + G_- X_-$$

$$\alpha(\omega) = e^{2i\omega B} \frac{G_-(\omega)}{G_+(\omega)}$$

$$f^{(\pm)}(\omega) = \mp \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{f(\omega')}{\omega - \omega' \mp ie}$$

$$0 = (G_- r_+)^{(-)} + (\alpha(G_+ X_+))^{(-)} + G_- X_-$$

$$X_+(\omega) = e^{-i\omega B} \tilde{R}(\omega) = X_-(-\omega)$$

$$\frac{\chi_+}{G_+} = (G_- r_+)^{(+)} + (\alpha(G_+ X_+))^{(+)} \rightarrow \mathcal{O}_{1,1} = \frac{e^B}{2\pi} \chi_+(i)$$

but where is the trans-series ?

Trans-series from Wiener-Hopf

after field redefinition and contour deformation ($N > 3$)

$$q_n(2i\xi) + \frac{i}{2\pi} \int_C \frac{\alpha(2i\xi') q_n(2i\xi')}{\xi + \xi'} d\xi' = \frac{1}{n - 2\xi}$$

$$q_n(ivx) = Q_n(x)$$

- Marino, Miravittlas, Reis, *JHEP* 08 (2022) 279 • [2111.11951](https://arxiv.org/abs/2111.11951)

$$Q_n(x) + 2i \sum_{l=1}^{\infty} \frac{H_l q_{n,l}}{vx + 2l\xi_0} e^{-4l\xi_0 B} + \frac{1}{\pi} \int \frac{e^{-2Bvy} \beta(vy/2) Q_n(y)}{x + y} dy = \frac{1}{n - 2vx}$$

$$q_{n,l} = q_n(2il\xi_0)$$

$$e^{-2Bvy} \beta(vy/2) = e^{-y} \mathcal{A}(y)$$

running coupling

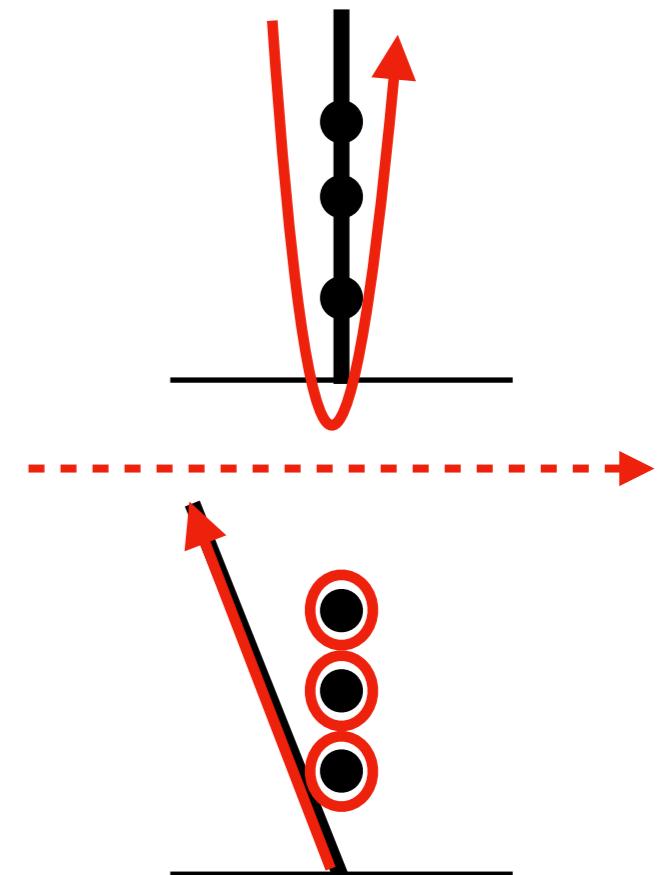
$$2B = \frac{1}{v} + (2\Delta - 1)\ln v + L$$

power series in v

$$q_{n,s} - 2i \sum_{l=1}^{\infty} H_l q_{n,l} e^{-4Bl\xi_0} A_{-2l,-2s} = A_{n,-2s}$$

$$\mathcal{O}_{nm} = \frac{e^{(n+m)B}}{4\pi} G_+(im) G_+(in) W_{n,m}$$

$$W_{n,m} = \frac{1}{n+m} + 2i \sum_{l=1}^{\infty} \frac{H_l q_{n,\kappa_l} e^{-4B\xi_0 l}}{m - \kappa_l} + \frac{v}{\pi} \int_{C_+} \frac{e^{-x} \mathcal{A}(x) Q_n(x)}{m - vx} dx = A_{n,m} + \dots$$



$$P_\alpha(x) + \int_{C_+} \frac{e^{-y} \mathcal{A}(y) P_\alpha(y)}{x + y} \frac{dy}{\pi} = \frac{1}{\alpha - vx}$$

$$A_{\alpha,\beta} = \frac{1}{\alpha + \beta} + \langle P_\alpha \rangle_\beta$$

$$\langle Q \rangle_\beta = \int_{C_+} \frac{e^{-x} \mathcal{A}(x) Q(x)}{\beta - vx} \frac{v dx}{\pi}$$

Full trans-series solution in O(4)

$$S_l = \frac{((2l-1)!!)^2}{2^{2l-1} l!(l-1)!}$$

$$W_{1,1} = A_{1,1} + M e^{-2B} + \sum_{l_1, l_2, \dots} e^{-4(l_1 + l_2 + \dots)B} i S_{l_1} i S_{l_2} \dots A_{1,-2l_1} A_{-2l_1,-2l_2} \dots A_{-l_k,1}$$

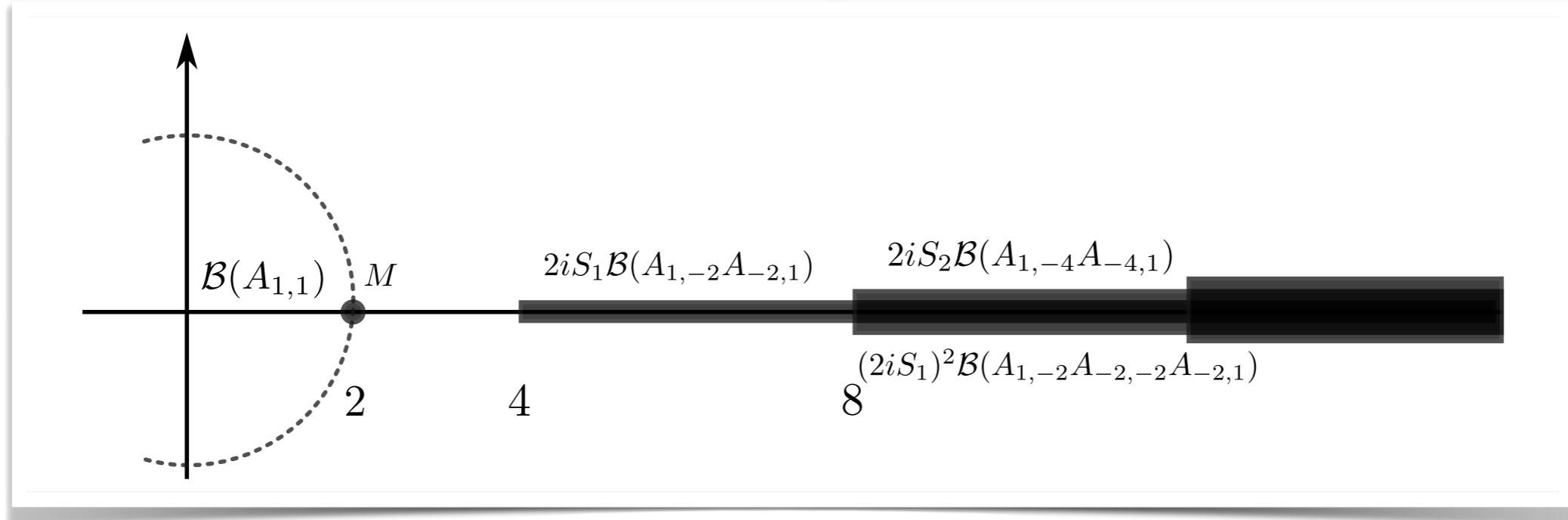
$$M = -2i$$

from the differential equations $A_{1,1} \rightarrow A_{n,m}$

$$A_{n,m} = \frac{1}{m+n} + \frac{\nu}{4mn} + \frac{\nu^2(20\gamma mn + 9m + 9n)}{32m^2n^2} + \frac{\nu^3 \left(m^2 (640\gamma^2 n^2 + 636\gamma n + 225) + 6mn(106\gamma n + 39) + 225n^2 \right)}{384m^3n^3} + O(\nu^4)$$

$$\nu = \frac{1}{2B}$$

$$W_{1,1} = A_{1,1} + M e^{-2B} + i e^{-4B} S_2 A_{1,-2}^2 + e^{-8B} ((i S_2)^2 A_{1,-2}^2 A_{-2,-2} + i S_4 A_{1,-4}^2) + \dots$$



$$\mathcal{O}_{11} = \frac{e^{2B}}{4\pi} G_+ (i)^2 W_{1,1}$$

uniquely defined and free of ambiguities, agrees with TBA

Conclusions

- The integrable description enabled to calculate high number of perturbative coefficient with high precision in the O(3) and O(4) models
- The asymptotic analysis of the perturbative coefficients revealed the analytic structure on the Borel plane with poles and cuts.
- The various alien derivatives with the median resummation provided a trans-series ansatz, whose leading terms matched perfectly with the numerical solution of the TBA equation in O(4)
- However, it failed to describe the leading real deviation from TBA in the O(3). This might be related to instantons!
- By expanding the integral equation using the Wiener-Hopf method, a trans-series form can be derived and systematically calculated, which matches in the O(3) model with the numerical solutions of the TBA equation
- The full trans-series solution is determined in terms of the perturbative $A_{n,m}$ basis, which can be explicitly calculated.
- The perturbative part completely determines all the non-perturbative corrections in the O(N>3) models but not in O(3), which might be related to an instanton saddle point