

Can we get non-perturbative
information from the
perturbative coefficients?

$$f(\alpha) = c_0 + c_1\alpha + \dots + c_n\alpha^n + \dots$$

$$+ e^{-8/\alpha} (d_0 + d_1\alpha + d_2\alpha^2 + \dots)$$

Knowing “enough” c-coefficients, can we get the d-s?

Yes!

From perturbative to non-perturbative in the O(4) sigma model

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based on 2011.12254 and 2011.09897

Plan

- Definition of the 2D O(N) models in a magnetic field
- Lagrangean perturbation theory
- Integrable description, TBA
- Expansion of the TBA, perturbative coefficients
- Asymptotic behaviour, the Borel plane
- Analytic structure on the Borel plane
- median resummation and non-perturbative contributions
- resurgence and trans-series
- Conclusions

Definition of the model

4 scalar fields in 2D living on the unit sphere

$$\Phi_1^2 + \dots + \Phi_4^2 = 1$$

magnetic field is coupled the conserved O(N) charge

$$\mathcal{L} = \frac{1}{2\lambda^2} \left\{ \partial_\mu \Phi_i \partial^\mu \Phi_i + 2ih(\Phi_1 \partial_0 \Phi_2 - \Phi_2 \partial_0 \Phi_1) + h^2(\Phi_3^2 + \Phi_4^2 - 1) \right\}$$

bare coupling

Euclidean Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 - hQ_{12}$$

Perturbation theory

$$\Phi_1^2 = 1 - \lambda^2(\varphi_2^2 + \varphi_3^2 + \varphi_4^2)$$

$$\lambda \varphi_i = \Phi_i$$

$$e^{-V\mathcal{F}(h)} = \int \mathcal{D}^3[\varphi] e^{-\int d^Dx \mathcal{L}(x)}$$

$$D = 2 - \epsilon$$

dimensional regularisation

Legendre transformation

$$\rho = -\partial\mathcal{F}/\partial h,$$

density

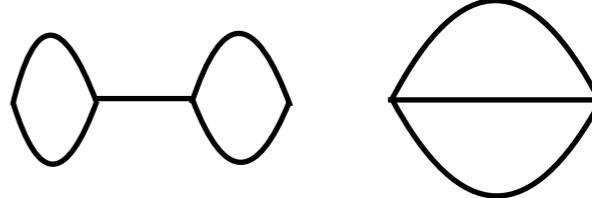
$$\epsilon(\rho) = \mathcal{F}(h) - \mathcal{F}(0) + \rho h$$

groundstate energy

Standard perturbation theory

$$\mathcal{F}(h) - \mathcal{F}(0) = -\frac{h^2}{2\lambda^2} + \frac{N-2}{4\pi} h^{2-\varepsilon} \left\{ \frac{1}{\varepsilon} + \frac{\gamma}{2} + \frac{1}{2} \right\} + \lambda^2 \frac{N-2}{16\pi^2} h^{2-2\varepsilon} \left\{ \frac{1}{\varepsilon} + \gamma + \frac{1}{2} \right\}$$

Bajnok, Balog, Basso, Korchemsky, Palla, *Nucl.Phys.B* 811 (2009) 438 , [0809.4952](#)



renormalized coupling

$$\lambda^2 = (\mu e^{\frac{\gamma}{2}})^{\varepsilon} Z_1 \tilde{g}^2 \quad \mu \frac{d\tilde{g}}{d\mu} = \beta(\tilde{g}) = -\beta_0 \tilde{g}^3 - \beta_1 \tilde{g}^5 + \dots$$

$$\mathcal{F}(h) - \mathcal{F}(0) = -\frac{h^2}{2} \left\{ \frac{1}{\tilde{g}^2} - 2\beta_0 \left(\ln \frac{\mu}{h} + \frac{1}{2} \right) - 2\beta_1 \tilde{g}^2 \left(\ln \frac{\mu}{h} + \frac{1}{4} \right) + O(\tilde{g}^4) \right\}$$

RG invariant dynamically generated scale

$$\Lambda = \mu e^{-\int_{\tilde{g}}^{\tilde{g}} \frac{dg}{\beta(g)}} = \mu e^{-\frac{1}{2\beta_0 \tilde{g}^2}} \tilde{g}^{-\beta_1/\beta_0^2} \left[1 + \frac{1}{2\beta_0} \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) \tilde{g}^2 + \dots \right] \quad \Delta = \frac{1}{N-2}$$

running coupling

$$\frac{1}{\tilde{\alpha}} + \Delta \ln \tilde{\alpha} = \ln \frac{h}{\Lambda_{\overline{MS}}}$$

$$\mathcal{F}(h) - \mathcal{F}(0) = -\beta_0 h^2 \left\{ \frac{1}{\tilde{\alpha}} - \frac{1}{2} - \frac{\Delta \tilde{\alpha}}{2} + O(\tilde{\alpha}^2) \right\}$$

After Legendre transformation

$$\frac{1}{\alpha} + (\Delta - 1) \ln \alpha = \ln \frac{\rho}{2\beta_0 \Lambda_{\overline{MS}}}$$

$$\epsilon(\rho) = \rho^2 \pi \Delta \left\{ \alpha + \frac{\alpha^2}{2} + \Delta \frac{\alpha^3}{2} + O(\alpha^4) \right\}$$

Integrable description

massive particles in the vector representation

$$\mathcal{H} = \mathcal{H}_0 - h Q_{12}$$

particles negatively charged under Q_{12}
condense into the vacuum

$$E_{\pm} = m \cosh \theta \pm h$$

multiparticle state on the circle
momentum quantization

$$S(\theta) = - \frac{\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi}) \Gamma(\Delta - \frac{i\theta}{2\pi}) \Gamma(1 + \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} + \Delta + \frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi}) \Gamma(\Delta + \frac{i\theta}{2\pi}) \Gamma(1 - \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} + \Delta - \frac{i\theta}{2\pi})}$$

Thermodynamic Bethe Ansatz: TBA

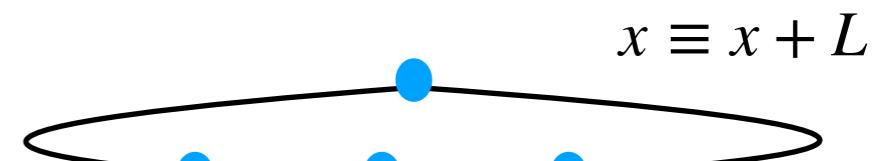
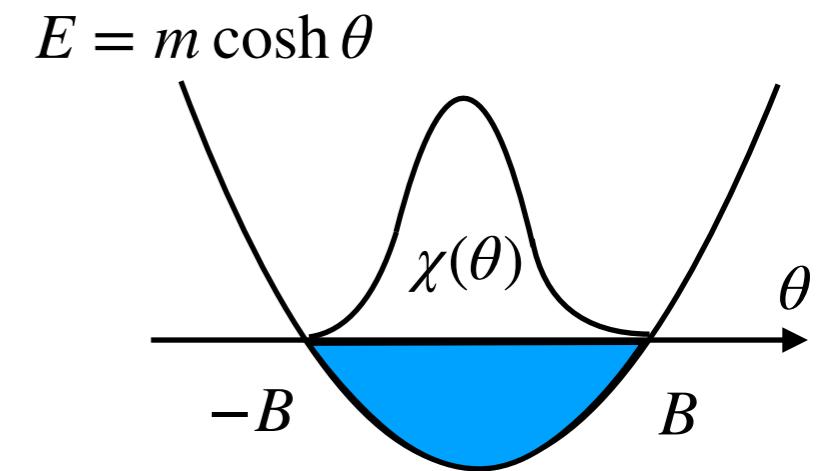
$$\chi(\theta) - \int_{-B}^B \frac{d\theta'}{2\pi} K(\theta - \theta') \chi(\theta') = m \cosh \theta$$

density

$$\rho = \int_{-B}^B \frac{d\theta}{2\pi} \chi(\theta)$$

ground state energy

$$\epsilon = m \int_{-B}^B \frac{d\theta}{2\pi} \cosh \theta \chi(\theta)$$



$$e^{imL \sinh \theta_j} \prod_k S(\theta_j - \theta_k) = 1$$

$$2\pi K(\theta) = - 2\pi i \partial_\theta \log S(\theta)$$

magnetic field

$$h(B) = \partial_\rho \epsilon(\rho) = \frac{\partial \epsilon}{\partial B} / \frac{\partial \rho}{\partial B}$$

Expansion of the TBA

Volin, Phys. Rev. D 81 (2010) 105008 · e-Print: 0904.2744

The resolvent

$$R(\theta) = \int_{-B}^B d\theta' \frac{\chi(\theta')}{\theta - \theta'}$$

its Laplace transform

$$z = 2(\theta - B)$$

$$\hat{R}(s) = \int_{-i\infty+0}^{i\infty+0} \frac{dz}{2\pi i} e^{sz} R(B + z/2)$$

density: residue at ∞

perturbative expansion in $1/B$

$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2} \log B\alpha = \log 2\hat{\rho}$$

energy: $\frac{\epsilon}{m} = \int_{-B}^B \cosh \theta \chi(\theta) \frac{d\theta}{2\pi} = \frac{e^B}{4\pi} \hat{R}(1/2)$

$$R(\theta) = 2A\sqrt{B} \sum_{n,m=0}^{\infty} \frac{c_{n,m}}{B^{m-n}(\theta^2 - B^2)^{n+1/2}}$$

$$\chi(\theta)$$



Wiener-Hopf

$$\hat{R}(s) = \frac{A}{\sqrt{s}} \frac{\Gamma(1+s)}{\Gamma(\frac{1}{2}+s)} \left(\frac{1}{s+\frac{1}{2}} + \frac{1}{Bs} \sum_{n,m=0}^{\infty} \frac{Q_{n,m}}{B^{n+m}s^n} \right)$$

$$A = \frac{me^B\sqrt{\pi}}{2\sqrt{2}}$$

$$\rho = A \frac{\sqrt{B}}{\pi} \hat{\rho} = A \frac{\sqrt{B}}{\pi} \sum_{m=0}^{\infty} c_{0,m} B^{-m}$$

$$\epsilon = \frac{me^B A}{4\sqrt{2\pi}} (1 + \sum_{k=0}^{\infty} \epsilon_k B^{-k-1}) \quad \epsilon_k = \sum_{j=0}^k 2^{j+1} Q_{j,k-j}$$

Perturbative coefficients

density	$\hat{\rho}(B) = 1 + \sum_{n=1}^{\infty} \frac{u_n}{B^n}$	$u_1 = -\frac{3}{8} + \frac{a}{2}$	$u_2 = -\frac{15}{128} + \frac{3a}{16} - \frac{a^2}{8}$	$u_3 = \frac{3\zeta_3}{64} + \frac{a^3}{16} - \frac{9a^2}{64} + \frac{45a}{256} - \frac{105}{1024}$
			$a = \ln 2$	odd zeta functions

energy	$\hat{\epsilon}(B) = 1 + \sum_{n=1}^{\infty} \frac{\xi_n}{B^n}$	$\left\{ \frac{1}{4}, \frac{9}{32} - \frac{a}{4}, \frac{a^2}{4} - \frac{9a}{16} + \frac{57}{128}, -\frac{a^3}{4} + \frac{27a^2}{32} - \frac{171a}{128} - \frac{27\zeta_3}{256} + \frac{1875}{2048}, \dots \right\}$
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Volin, *Phys.Rev.D* 81 (2010) 105008 • e-Print: [0904.2744](#)

Marino, Reis, *JHEP* 04 (2020) 160 • e-Print: [1909.12134](#)

22 coefficients

44 coefficients

50 coefficients

free energy in the running coupling

$$B = \frac{1}{\alpha} + \frac{1}{2} - \ln 2 + \frac{\alpha}{8} + \frac{13 - 18\zeta_3}{384}\alpha^3 + \dots$$

$$f(\alpha) = \frac{\epsilon}{\rho^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \chi_n \alpha^n = \frac{\pi}{2} \left(1 + \frac{\alpha}{2} + \frac{\alpha^2}{4} + \frac{10 - 3\zeta_3}{32}\alpha^3 + \chi_5\alpha^4 + \dots \right)$$

$$\left\{ 1, \frac{1}{2}, \frac{1}{4}, \frac{5}{16} - \frac{3\zeta_3}{32}, \frac{53}{96} - \frac{9\zeta_3}{64}, -\frac{189\zeta_3}{512} - \frac{405\zeta_5}{2048} + \frac{487}{384}, \dots \right\}$$

Hasenfratz, Maggiore, Niedermayer, *Phys.Lett.B* 245 (1990) 522

Comparing ordinary perturbation theory in $\frac{h}{\Lambda}$ to expansion of TBA in $\frac{h}{m}$
 relation between mass and scale can be obtained $m/\Lambda = (8/e)^{\Delta}/\Gamma(1 + \Delta)$

Numerical data

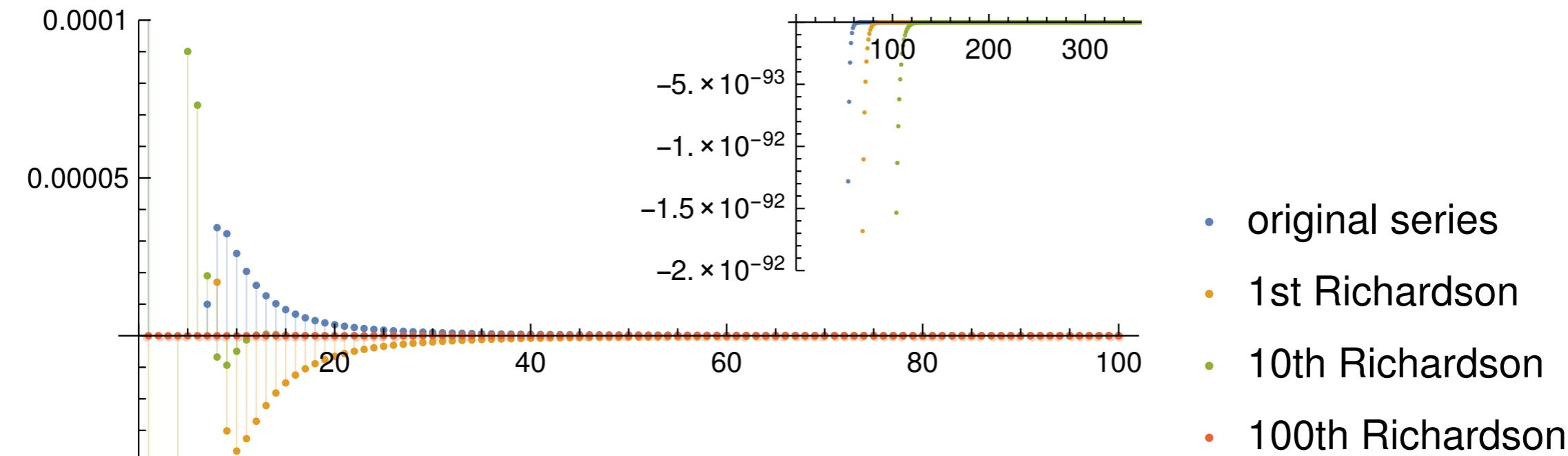
density $\hat{\rho}(B) = 1 + \sum_{n=1}^{\infty} \frac{u_n}{B^n}$ 2000 coefficients for 7000 digits

energy $\hat{e}(B) = 1 + \sum_{n=1}^{\infty} \frac{\xi_n}{B^n}$ 2000 coefficients for 7000 digits

free energy $\hat{f}(\alpha) = \frac{\hat{e}}{\hat{\rho}^2} = \sum_{n=1}^{\infty} \chi_n \alpha^n$ 1400 coefficients for 4000 digits

how constant is the asymptotics?

$$c_n = \frac{\chi_{n+2} 2^{n+1}}{\Gamma(n+1)} = p^+ + (-1)^n p^- + \frac{1}{n} (\dots)$$



$$a_n = c_{2n} + c_{2n-1} = A + \frac{A_1}{n} + \dots$$



$$(n+1)a_{n+1} - na_n = A + \frac{B}{n^2} + \dots$$

Asymptotic behaviour

perturbative coefficient grow factorially:
how to give meaning to the series?

$$c_n = \frac{\chi_{n+2} 2^{n+1}}{\Gamma(n+1)}$$

Borel function

$$B(t) = \sum_{n=1}^{\infty} c_n t^n$$

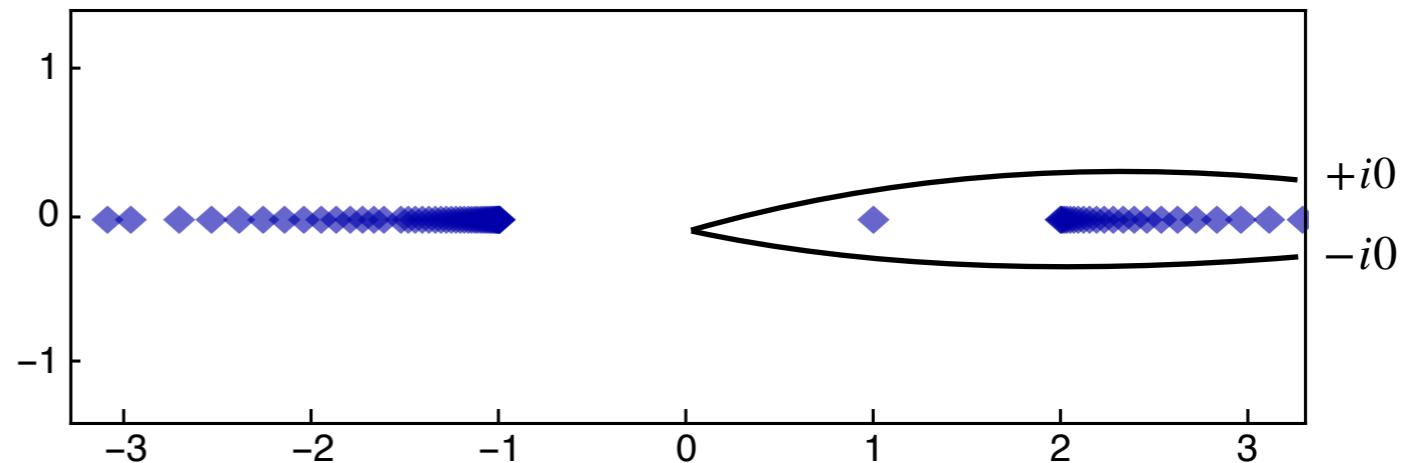
path integral

$$e^{-V\mathcal{F}(h)} = \int \mathcal{D}^3[\varphi] e^{-S[\varphi]} = \int_0^{\infty} dt e^{-\frac{2t}{\alpha}} \int_{S[\varphi]=\frac{2t}{\alpha}} \mathcal{D}^3[\varphi] e^{-S[\varphi]}$$

convergence radius 1

Pade approximant

$$B(t) \approx \frac{\sum_{i=1}^n \beta_i t^i}{1 + \sum_{j=1}^m \gamma_j t^j}$$



Ambiguity from the pole

$$\frac{i\pi^2 \alpha \text{res}_1 B(t)}{4} e^{-\frac{2}{\alpha}}$$

$$f^{(\pm)}(\alpha) = \frac{\pi}{2} \left[\chi_1 \alpha + \chi_2 \alpha^2 + \alpha \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}} B(t) dt \right]$$

Better approximation —————→

Conformal mapping

Conformal mapping vs numerical solution



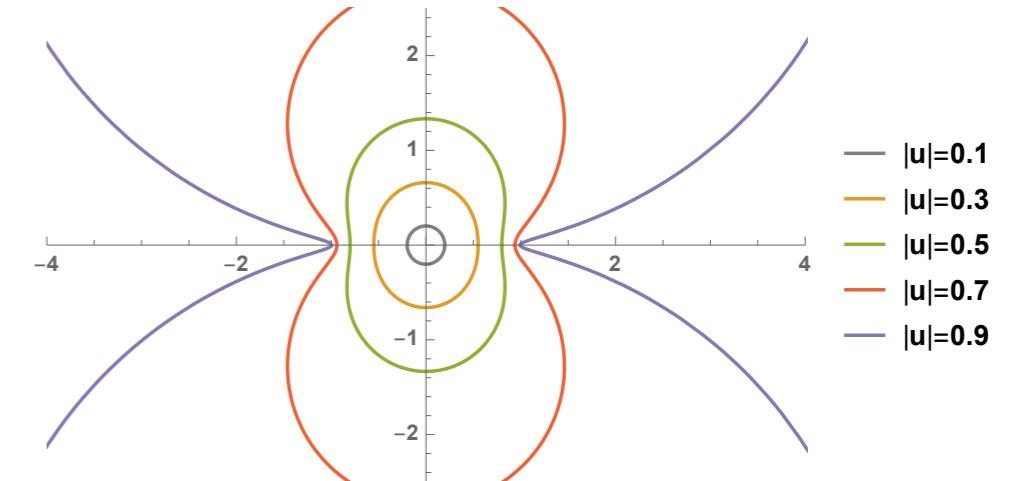
Conformal mapping maps the t-plane to the unit u-disk

$$u(t) = \frac{1 - \sqrt{1 - t^2}}{t}$$

$$\tilde{B}(u) = \sum_{n=1}^{\infty} b_n u^n$$

$$\tilde{B}(t) = \sum_{n=1}^{\infty} b_n u(t)^n$$

$$f^{(\pm)}(\alpha) = \frac{\pi}{2} \left[\chi_1 \alpha + \chi_2 \alpha^2 + \alpha \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}} B(t) dt \right]$$



**Compare to numerical solution
in a Chebyshev basis**

$$\chi(\theta) = \sum_{j=1}^{(n_c+1)/2} s_j T_{2j-2}(\theta/B), \quad T_n(x) = \cos(n \arccos x)$$

precision=30 digits

$$\operatorname{Im}(f^{(+)}(\alpha)) = \alpha c_0 e^{-2/\alpha} + \alpha e^{-4/\alpha} (c_1 + c_2 \alpha + \dots)$$

$$c_0 = 1.70067333(1)$$

$$c_1 = -1.70067333(1)$$

$$c_2 = 0.637752(1)$$

$$c_3 = -0.1727(1)$$

$$\operatorname{Re}(f^{(+)}(\alpha)) = f_{\text{TBA}}(\alpha) + \alpha e^{-8/\alpha} (d_0 + d_2 \alpha + \dots)$$

$$d_1 = -0.9206(1)$$

$$d_2 = 0.575(3)$$

Asymptotic analysis

$$\Psi(z) = 1 + \sum_{n=1}^{\infty} s_n/z^n \quad z = 2B \quad c_n = s_{n+1}/n! \quad B(t) = \sum_{n=0}^{\infty} c_n t^n$$

Asymptotic large n behaviour

$$c_n = (-1)^n \left(p^- + \frac{p_0^-}{n} + \frac{p_1^-}{n(n-1)} + \dots \right) + \left(p^+ + \frac{p_0^+}{n} + \frac{p_1^+}{n(n-1)} + \dots \right) + 2^{-n} \left(q^+ + \frac{q_0^+}{n} + \frac{q_1^+}{n(n-1)} + \dots \right)$$

$$\log(1+t) \left[-p_0^- + p_1^-(1+t) + \dots \right] + \frac{p^-}{1+t}$$

$\log(1+t)\Phi(1+t)$

alien derivative

$$\Delta_{\pm 1}\Psi(z) = \mp i2\pi \left\{ p^\pm \pm \sum_{m=0}^{\infty} \frac{(\pm 1)^m p_m^\pm}{z^{m+1}} \right\}$$

Dorigoni, *Annals Phys.* 409 (2019) 167914 • e-Print: [1411.3585](https://arxiv.org/abs/1411.3585)

Aniceto, Basar, Schiappa, *Phys.Rept.* 809 (2019) 1-135 • e-Print: [1802.10441](https://arxiv.org/abs/1802.10441)

Asymptotics for $\hat{f}(\alpha)$

numerical fitting

2.80308535473939142809960724226717498614747943851074832268840733301275 7308679469635279683810414002887

$$p^- = -\frac{e}{8\pi}$$

for 150 digits

$$p_0^- = 0$$

for 147 digits

$$p_1^- = \frac{e}{4\pi}$$

for 144 digits

$$p_2^- = \frac{e}{4\pi} \left(-\frac{1}{2} - \frac{3}{4}\zeta_3 \right)$$

<http://wayback.cecm.sfu.ca/projects/EZFace/>

$$\Delta_{-1}\hat{f}$$

$$p^+ = \frac{8}{e\pi}$$

$$p_0^+ = 0$$

$$p_1^+ = 0$$

$$q^- = \frac{16}{e^2\pi} \quad \text{for 80 digits}$$

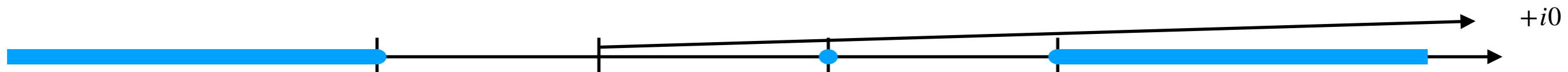
$$q_0^- = \frac{16}{e^2\pi} \left(-\frac{3}{4} \right)$$

$$q_0^- = \frac{16}{e^2\pi} \left(\frac{13}{32} \right)$$

$$q_0^- = \frac{16}{e^2\pi} \left(-\frac{99}{256} + \frac{3}{8}\zeta_3 \right)$$

$$\Delta_1\hat{f}$$

$$\Delta_2\hat{f}$$



imaginary ambiguity

$$f^{(+)}(\alpha) = \frac{\pi}{2} \left[\alpha \int_0^{\infty+i0} e^{-\frac{2t}{\alpha}} B(t) dt \right]$$

$$\Im m(f^{(+)}(\alpha)) = \frac{4\pi}{e^2} \alpha e^{-2/\alpha} + \alpha e^{-4/\alpha} \left(-\frac{4\pi}{e^2} + \frac{3\pi}{2e^2} \alpha - \frac{13\pi}{32e^2} \alpha^2 + \dots \right)$$

$$c_0 = 1.70067333$$

$$c_1 = -1.70067333(1)$$

$$c_2 = 0.637752(1)$$

$$c_3 = -0.1727(1)$$

real ambiguity???

Median resummation and Stokes automorphism

$$S_{\pm}(f) = f^{(\pm)} = \chi_1 + \alpha \chi_2 + \int_0^{\infty \pm i0} e^{-tx} B(t) dt$$

$x = \frac{2}{\alpha}$

$$S_+(f) - S_-(f) = -S_+ \left(e^{-x} \Delta_1 f + e^{-2x} \Delta_2 f + \dots + \frac{e^{-2x}}{2} \Delta_1^2 f + \dots \right)$$

$$S_+(f) = S_-(\mathfrak{S}f) \quad ; \quad S_-(f) = S_+(\mathfrak{S}^{-1}f)$$

cut of the cut of the cut...

$$\mathfrak{S} = \exp \left\{ - \sum_{n=1}^{\infty} e^{-nx} \Delta_n \right\}$$

Median resummation

$$S_{\text{med}}(f) = S_-(\mathfrak{S}^{\frac{1}{2}}f) = S_+(\mathfrak{S}^{-\frac{1}{2}}f) = S_+(e^{\frac{1}{2}} \sum e^{-nx} \Delta_n f)$$

$$S_{\text{med}}(f) = S_+ \left(f + \frac{e^{-x}}{2} \Delta_1 f + \frac{e^{-2x}}{2} \Delta_2 f + \dots + \frac{e^{-4x}}{8} \Delta_1 \Delta_3 f + \frac{e^{-4x}}{8} \Delta_2^2 f + \dots \right)$$

$$\Delta_1 f = -\frac{16i}{e^2} \quad \Delta_2 f = \frac{16i}{e^2} \left(1 - \frac{3}{4x} + \frac{13}{32x^2} - \left(\frac{99}{256} - \frac{3}{8} \zeta_3 \right) \frac{1}{x^3} + \dots \right)$$

We need $\Delta_2 \Delta_2 f$

need the asymptotics of $\Delta_2 f$

$$S_+(\Delta_2 f) - S_-(\Delta_2 f) = -S_+ (e^{-2x} \Delta_2 \Delta_2 f) + \dots$$

Spoiler $S_{\text{med}}(f) = \Re(S_+(f)) + \frac{32}{e^4} e^{-8/\alpha} \left(1 - \frac{5\alpha}{8} + \dots \right)$ $d_1 = 0.58607$ $\frac{d_2}{d_1} = -0.6246$

Resurgence in 1/B

Alien derivatives from asymptotics

$$\hat{\rho}(B)$$



$$\Delta_{-1}\hat{\rho} = i\hat{\epsilon}\hat{\rho},$$

$$\Delta_1\hat{\rho} = 0,$$

$$\Delta_2\hat{\rho} = iR/2,$$

$$R = 1 + \sum_{n=1}^{\infty} r_n/z^n.$$

$$\hat{\epsilon}(B)$$



$$\Delta_{-1}\hat{\epsilon} = i\hat{\epsilon}^2,$$

$$\Delta_1\hat{\epsilon} = -4i.$$

$$\Delta_2\hat{\epsilon} = 2iE,$$

$$E = 1 + \sum_{n=1}^{\infty} e_n/z^n$$

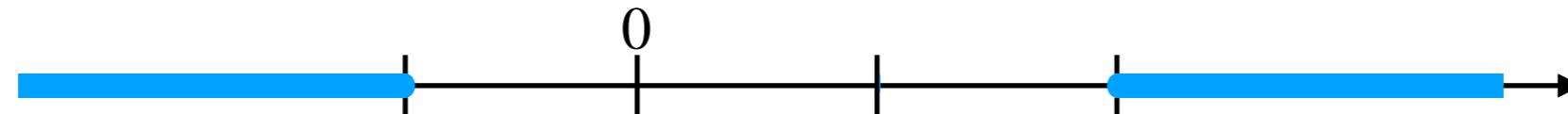
asymptotics of the asymptotics: analytic structure of the alien derivatives

$$R(B)$$

$$r_1 = 1/2 + a$$

$$r_2 = -a/2 - a^2/2$$

$$r_3 = \frac{21}{64} + \frac{3}{4}a^2 + \frac{a^3}{2} + \frac{3}{8}\zeta_3$$



$$\Delta_{-1}R = i(4\hat{\rho}E + \hat{\epsilon}R)$$

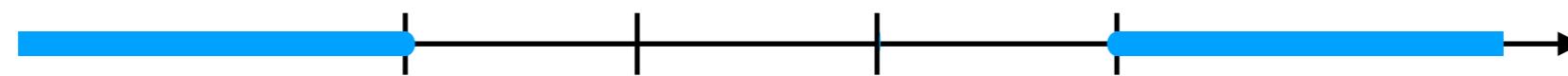
$$\Delta_1R = 0$$

$$\Delta_2R = -i/2(1 + (\frac{1}{4}+a)/z + \dots)$$

$$\Delta_{-1}E = 2i\hat{\epsilon}E$$

$$\Delta_1E = 0$$

$$\Delta_2E = -i/2(1 + 3/8z^2 + \dots)$$



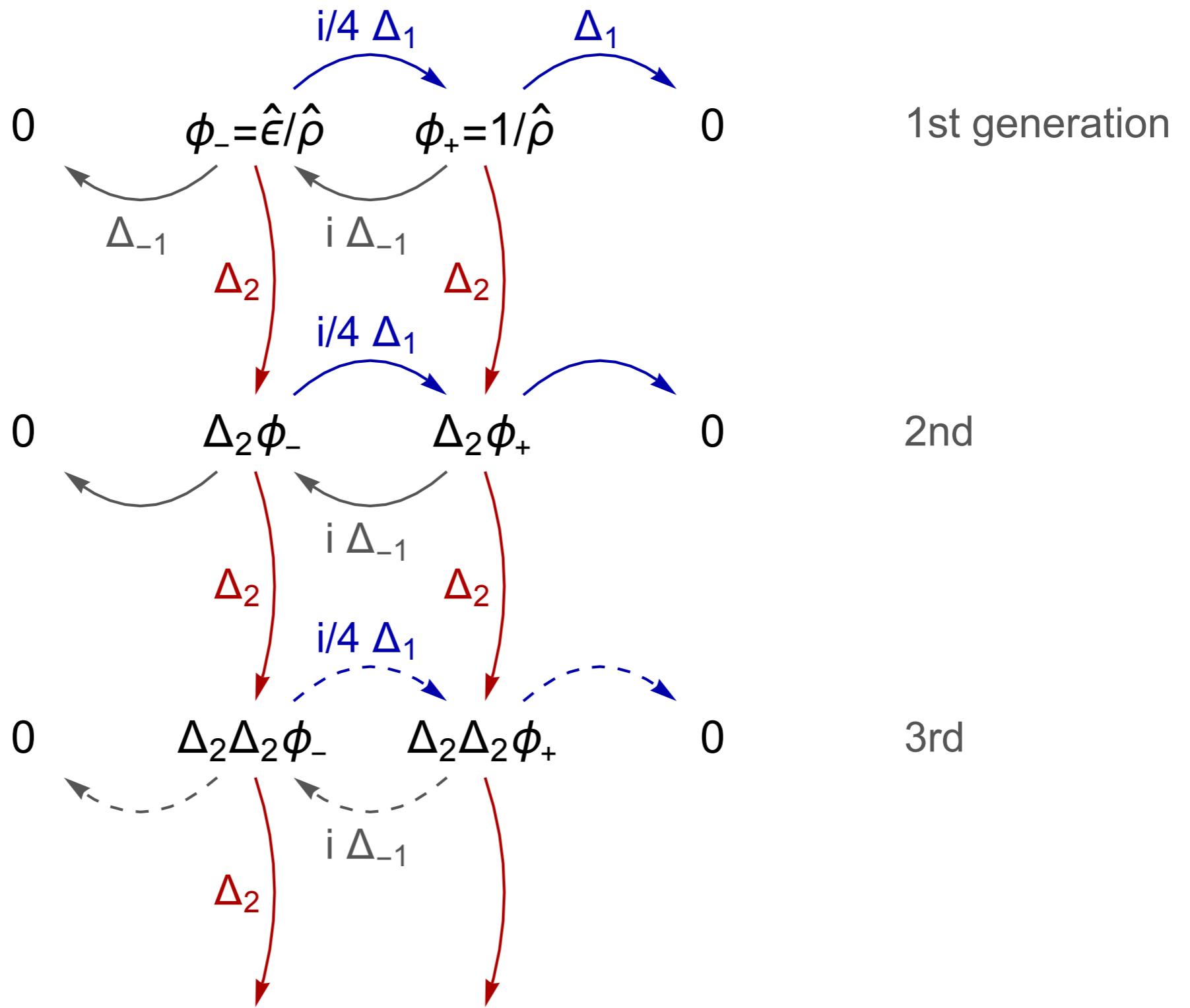
$$E(B)$$

$$e_1 = 1/4$$

$$e_2 = 5/32 - a/2$$

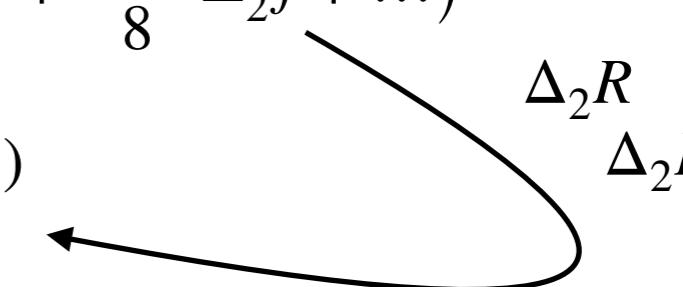
$$e_3 = \frac{57}{128} - \frac{5}{8}a + a^2 \quad + 60 \text{ terms}$$

Resurgence structure

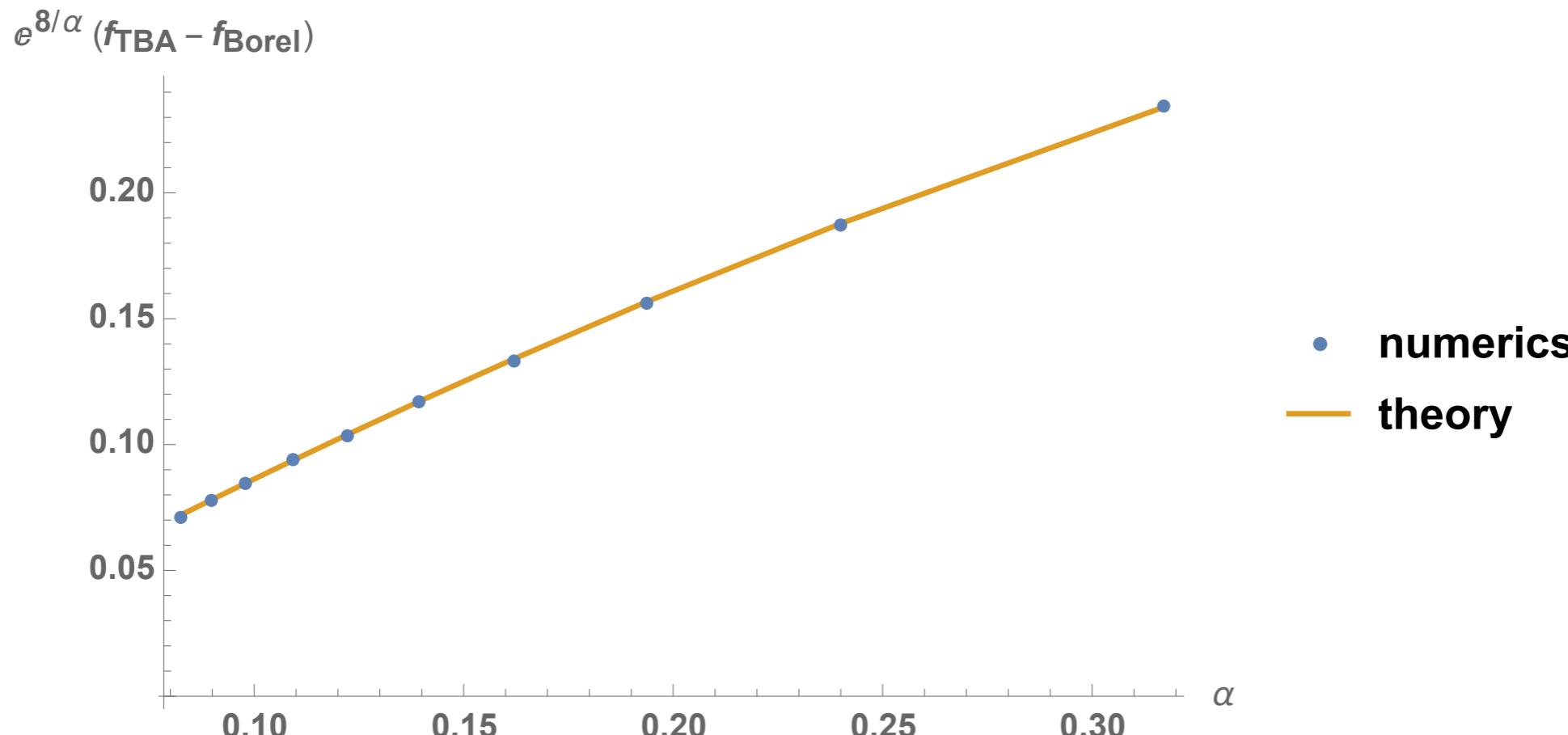


Comparison with TBA

Median resummation

$$S_{\text{med}}(f) = S_+ \left(f + \frac{e^{-x}}{2} \Delta_1 f + \frac{e^{-2x}}{2} \Delta_2 f + \dots + \frac{e^{-4x}}{8} \Delta_1 \Delta_3 f + \frac{e^{-4x}}{8} \Delta_2^2 f + \dots \right)$$
$$f_{\text{Borel}} = S_{\text{med}}(f) = \Re(S_+(f)) + \frac{32}{e^4} e^{-8/\alpha} \left(1 - \frac{5\alpha}{8} + \dots \right)$$
$$d_1 = 0.58607 \quad \frac{d_2}{d_1} = -0.6246$$
$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2} \log B\alpha = \log 2\hat{\rho}$$


We compare to the numerical solution of TBA



Trans-series

Analytic structure of the free energy on the Borel plane



The expansion of the physical observable is a trans-series

$$f_{\text{TBA}} = \sum_{m=0}^{\infty} e^{-\frac{2}{\alpha}m} \sum_{n=1}^{\infty} \chi_n^{(m)} \alpha^{n-1}$$

$$\chi_n^{(0)} = \chi_n \quad \chi_n^{(1)} = -\frac{16i}{e^2} \delta_{n,0} \quad \chi_n^{(2)}$$

perturbative coefficients

$$\Delta_1 f = -\frac{16i}{e^2} \quad \Delta_2 f = \frac{16i}{e^2} \left(1 - \frac{3}{4x} + \frac{13}{32x^2} - \left(\frac{99}{256} - \frac{3}{8}\zeta_3 \right) \frac{1}{x^3} + \dots \right)$$

Conclusions

- The integrable description enabled to calculate 2000 perturbative coefficient with high precision
- The asymptotic analysis of the perturbative coefficients revealed the analytic structure on the Borel plane with poles and cuts. The leading cuts showed a nice resurgence structure
- The various alien derivatives with the median resummation provided a trans-series ansatz, whose leading terms matched perfectly with the numerical solution of the TBA equation
- We recovered non-perturbative information from the perturbative series!

Some open problems

- Calculate analytically the numerically determined coefficients
- Explain the resurgence structure
- Derive the trans-series ansatz
- Extend for other $O(N)$ models ($O(3)$ has instantons)
- Investigate the large N behaviour

Marino, Mas, Reis, e-Print: [2102.03078](https://arxiv.org/abs/2102.03078)